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NLO evolution of 3-quark Wilson loop operator

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Abstract: The NLO evolution equation for the 3-quark Wilson loop operator was obtained. A composite 3-quark Wilson loop operator obeying a quasi-conformal evolution equation was constructed. The linearized quasi-conformal equation is presented.

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1 Introduction

This paper develops the Wilson line approach to the high energy scattering proposed in [1] to the case of the 3-quark Wilson loop (3QWL) operator in the next to leading order (NLO). The full NLO hierarchy of the Wilson lines and the JIMWLK hamiltonian equivalent to it are obtained in [2] and [3]. However, the NLO evolution of the color dipole operator in known for some time [4], [5], [6].

The 3QWL is a colorless operator which has a baryon structure $\varepsilon^{i'j'h'}\varepsilon_{ijh}U^i_{1i'}U^j_{2j'}U^h_{3h'}$. Its leading order (LO) linear evolution equation was studied in the C-odd case within the JIMWLK formalism and proved equivalent to the C-odd BKP equation [7]-[8] in [9] and its nonlinear evolution equation was derived within Wilson line approach [1] in [10]. The connected contribution to the kernel of the equation was calculated in [11]. In the momentum representation the evolution of this operator was first studied in [12], and the nonlinear equation was worked out in [13]. In the C-odd case the linear NLO evolution equation for the odderon Green function was obtained in [14].

In this paper we use the results of [2], [6] and [11] to construct the NLO evolution equation for the 3QWL operator. Then as in [5], we construct the composite 3QWL operator obeying the quasi-conformal evolution equation and give its linearized kernel.

The paper is organized as follows. The next section contains the definitions and necessary results. Section 3 presents the derivation of the NLO kernel for 3QWL operator. Section 4 describes the calculation of the quasi-conformal kernel for the composite 3QWL operator. Section 5 gives the linearized kernel. Section 6 lists the main results. Section 7 concludes the paper. Appendices comprise necessary technical details.

2 Definitions and necessary results

We use the following notation. We introduce the light cone vectors n_1 and n_2

$$n_1 = (1,0,0,1)\,, \quad n_2 = \frac{1}{2}(1,0,0,-1)\,, \quad n_1^+ = n_2^- = n_1 n_2 = 1 \tag{2.1}$$

and for any vector p we have

$$p^+ = p_- = pn_2 = \frac{1}{2}\left(p^0 + p^3\right), \qquad p_+ = p^- = pn_1 = p^0 - p^3, \tag{2.2}$$

$$p = p^+ n_1 + p^- n_2 + p_\perp, \qquad p^2 = 2p^+p^- - \vec{p}^{\,2}, \tag{2.3}$$

$$p\,k = p^\mu k_\mu = p^+k^- + p^-k^+ - \vec{p}\vec{k} = p_+k_- + p_-k_+ - \vec{p}\vec{k}. \tag{2.4}$$

We define the 3QWL operator as

$$B_{123} = \varepsilon^{i'j'h'}\varepsilon_{ijh} U_{1i'}^{i} U_{2j'}^{j} U_{3h'}^{h} = U_1 \cdot U_2 \cdot U_3, \tag{2.5}$$

where

$$U_i = U\left(\vec{r}_i, \eta\right) = Pe^{ig\int_{-\infty}^{+\infty} b_\eta^-(r^+,\vec{r})dr^+}, \tag{2.6}$$

and b_η^- is the external shock wave field built from only slow gluons

$$b_\eta^- = \int \frac{d^4p}{(2\pi)^4} e^{-ipz} b^-(p)\,\theta(e^\eta - p^+). \tag{2.7}$$

The index convention is $a_i^j b_j^k = (ab)_i^k$. The parameter η separates the slow gluons entering the Wilson lines from the fast ones in the impact factors. The field

$$b^\mu(r) = b^-(r^+,\vec{r}) n_2^\mu = \delta(r^+) b(\vec{r})\, n_2^\mu. \tag{2.8}$$

Therefore $\vec{r}_1, \vec{r}_2, \vec{r}_3$ are the coordinates of the quarks within the 3QWL, and we denote $\vec{r}_0, \vec{r}_4$ as the coordinates of the gluons. Hereafter we set $N_c = 3$ explicitly. Then one can use the $SU(3)$ identities

$$U_4^{ba} = 2tr(t^b U_4 t^a U_4^\dagger), \quad (t^a)_i^j (t^a)_k^l = \frac{1}{2}\delta_i^l \delta_k^j - \frac{1}{6}\delta_i^j \delta_k^l \tag{2.9}$$

to rewrite the result of [2] only through the Wilson lines in the fundamental representation. For the contribution of the the states with 2 gluons crossing the shockwave it reads

$$\langle K_{NLO} \otimes (U_1)_{i_1}^{i_3}(U_2)_{j_1}^{j_3}\rangle|_{2g} = -\frac{\alpha_s^2}{8\pi^4}\int d\vec{r}_0 d\vec{r}_4\; \mathbf{G}_{12}, \tag{2.10}$$

$$
\begin{aligned}
\mathbf{G}_{12} = G_1 &\left\{ \left(U_1 U_0^\dagger U_4\right)_{i_1}^{i_3} \left(U_0 U_4^\dagger U_2\right)_{j_1}^{j_3} + \left(U_4 U_0^\dagger U_1\right)_{i_1}^{i_3} \left(U_2 U_4^\dagger U_0\right)_{j_1}^{j_3} \right\} \\
&+ G_2 (U_1)_{i_1}^{i_3} \left\{ \left(U_0 U_4^\dagger U_2 U_0^\dagger U_4\right)_{j_1}^{j_3} + \left(U_4 U_0^\dagger U_2 U_4^\dagger U_0\right)_{j_1}^{j_3} \right\} \\
&+ G_3 (U_2)_{j_1}^{j_3} \left\{ \left(U_0 U_4^\dagger U_1 U_0^\dagger U_4\right)_{i_1}^{i_3} + \left(U_4 U_0^\dagger U_1 U_4^\dagger U_0\right)_{i_1}^{i_3} \right\} \\
&+ G_4 \left\{ (U_4)_{j_1}^{i_3} \left(U_1 U_0^\dagger U_2\right)_{i_1}^{j_3} + (U_4)_{i_1}^{j_3} \left(U_2 U_0^\dagger U_1\right)_{j_1}^{i_3} \right\} tr(U_0 U_4^\dagger) \\
&+ G_5 (U_2)_{j_1}^{j_3} (U_4)_{i_1}^{i_3} tr(U_0 U_4^\dagger) tr(U_0^\dagger U_1) \\
&+ G_6 \left((U_0)_{i_1}^{i_3} \left\{ \left(U_4 U_0^\dagger U_1 U_4^\dagger U_2\right)_{j_1}^{j_3} + \left(U_2 U_4^\dagger U_1 U_0^\dagger U_4\right)_{j_1}^{j_3} \right\} \right. \\
&- \left. \left\{ (U_4)_{j_1}^{i_3} \left(U_0 U_4^\dagger U_2\right)_{i_1}^{j_3} + (U_4)_{i_1}^{j_3} \left(U_2 U_4^\dagger U_0\right)_{j_1}^{i_3} \right\} tr(U_0^\dagger U_1) \right) \\
&+ G_7 (U_1)_{i_1}^{i_3} (U_4)_{j_1}^{j_3} tr(U_0 U_4^\dagger) tr(U_0^\dagger U_2) \\
&+ G_8 \left((U_4)_{j_1}^{j_3} \left\{ \left(U_0 U_4^\dagger U_2 U_0^\dagger U_1\right)_{i_1}^{i_3} + \left(U_1 U_0^\dagger U_2 U_4^\dagger U_0\right)_{i_1}^{i_3} \right\} \right. \\
&- \left. \left\{ (U_0)_{i_1}^{j_3} \left(U_4 U_0^\dagger U_1\right)_{j_1}^{i_3} + (U_0)_{j_1}^{i_3} \left(U_1 U_0^\dagger U_4\right)_{i_1}^{j_3} \right\} tr(U_4^\dagger U_2) \right). \qquad (2.11)
\end{aligned}
$$

We did not write the subtraction terms here since it would be easier to make the subtraction after the color convolution. The functions have the form

$$
\begin{aligned}
G_1 = &-\left(\frac{\vec{r}_{04}{}^2 - 2\vec{r}_{02}{}^2}{2\vec{r}_{02}{}^2 \vec{r}_{04}{}^2 \left(\vec{r}_{24}{}^2 - \vec{r}_{02}{}^2\right)} - \frac{\vec{r}_{04}{}^2 \vec{r}_{12}{}^2 + \vec{r}_{02}{}^2 \left(\vec{r}_{12}{}^2 - \vec{r}_{14}{}^2\right) + \left(\vec{r}_{01}{}^2 + \vec{r}_{02}{}^2 - \vec{r}_{12}{}^2\right) \vec{r}_{24}{}^2}{2\vec{r}_{01}{}^2 \vec{r}_{02}{}^2 \vec{r}_{04}{}^2 \vec{r}_{24}{}^2} \right. \\
&+ \left. \frac{1}{\vec{r}_{01}{}^2 \vec{r}_{24}{}^2 - \vec{r}_{02}{}^2 \vec{r}_{14}{}^2} \left[\frac{2\vec{r}_{12}{}^2}{\vec{r}_{04}{}^2} - \frac{\vec{r}_{12}{}^4}{2\vec{r}_{01}{}^2 \vec{r}_{24}{}^2} - \frac{\vec{r}_{14}{}^2}{\vec{r}_{04}{}^2} - \frac{\left(\vec{r}_{02}{}^2 - \vec{r}_{04}{}^2\right)\left(\vec{r}_{14}{}^2 - \vec{r}_{01}{}^2\right) \vec{r}_{24}{}^2}{\vec{r}_{04}{}^4 \left(\vec{r}_{24}{}^2 - \vec{r}_{02}{}^2\right)} \right] \right) \\
&\times \ln\left(\frac{\vec{r}_{02}{}^2}{\vec{r}_{24}{}^2}\right) - \frac{1}{2\vec{r}_{04}{}^4} + (0 \leftrightarrow 4, 1 \leftrightarrow 2). \qquad (2.12)
\end{aligned}
$$

$$
\begin{aligned}
G_2 = &\left(\frac{1}{\left(\vec{r}_{02}{}^2 - \vec{r}_{24}{}^2\right)} \left[\left(\frac{1}{\vec{r}_{04}{}^4} + \frac{1}{2\vec{r}_{02}{}^2 \vec{r}_{24}{}^2} \right) \frac{\left(\vec{r}_{02}{}^2 + \vec{r}_{24}{}^2\right)}{2} - \frac{2}{\vec{r}_{04}{}^2} \right] - \frac{\vec{r}_{02}{}^2 - \vec{r}_{24}{}^2}{4\vec{r}_{02}{}^2 \vec{r}_{04}{}^2 \vec{r}_{24}{}^2} \right) \\
&\times \ln\left(\frac{\vec{r}_{02}{}^2}{\vec{r}_{24}{}^2}\right) - \frac{1}{\vec{r}_{04}{}^4}. \qquad (2.13)
\end{aligned}
$$

$$
G_3 = G_2|_{1\leftrightarrow 2}. \qquad (2.14)
$$

$$
\begin{aligned}
G_4 = &\left(\frac{\left(\vec{r}_{02}{}^2 - \vec{r}_{24}{}^2\right)\left(\vec{r}_{02}{}^2 \vec{r}_{14}{}^2 - \vec{r}_{01}{}^2 \vec{r}_{24}{}^2\right)}{2\vec{r}_{01}{}^2 \vec{r}_{02}{}^2 \vec{r}_{14}{}^2 \vec{r}_{24}{}^2 \vec{r}_{04}{}^2} - \frac{\vec{r}_{02}{}^2 + \vec{r}_{24}{}^2}{2\vec{r}_{02}{}^2 \vec{r}_{24}{}^2 \vec{r}_{04}{}^2} + \frac{1}{2\vec{r}_{01}{}^2 \vec{r}_{14}{}^2} + \frac{\vec{r}_{12}{}^4}{2\vec{r}_{01}{}^2 \vec{r}_{02}{}^2 \vec{r}_{14}{}^2 \vec{r}_{24}{}^2} \right. \\
&+ \left. \left(\frac{\left(\vec{r}_{01}{}^2 - \vec{r}_{14}{}^2\right)\left(\vec{r}_{02}{}^2 - \vec{r}_{24}{}^2\right)}{2\vec{r}_{01}{}^2 \vec{r}_{02}{}^2 \vec{r}_{04}{}^2 \vec{r}_{14}{}^2 \vec{r}_{24}{}^2} - \frac{\vec{r}_{02}{}^2 + \vec{r}_{24}{}^2}{2\vec{r}_{01}{}^2 \vec{r}_{02}{}^2 \vec{r}_{14}{}^2 \vec{r}_{24}{}^2} \right) \vec{r}_{12}{}^2 \right) \ln\left(\frac{\vec{r}_{01}{}^2}{\vec{r}_{14}{}^2}\right) - G_1. \qquad (2.15)
\end{aligned}
$$

$$G_5=\frac{2}{\vec{r}_{04}{}^4}+\left(\frac{1}{\vec{r}_{01}{}^2-\vec{r}_{14}{}^2}\left[\frac{4}{\vec{r}_{04}{}^2}-\frac{\vec{r}_{01}{}^2+\vec{r}_{14}{}^2}{\vec{r}_{04}{}^4}-\frac{1}{\vec{r}_{01}{}^2}\right]-\frac{1}{\vec{r}_{01}{}^2\vec{r}_{04}{}^2}\right)\ln\left(\frac{\vec{r}_{01}{}^2}{\vec{r}_{14}{}^2}\right). \quad (2.16)$$

$$G_6=\left(\frac{\vec{r}_{12}{}^2-\vec{r}_{24}{}^2}{2\vec{r}_{01}{}^2\vec{r}_{14}{}^2\vec{r}_{24}{}^2}+\frac{\vec{r}_{14}{}^2\left(\vec{r}_{24}{}^2-\vec{r}_{02}{}^2\right)+\vec{r}_{01}{}^2\left(\vec{r}_{14}{}^2-\vec{r}_{12}{}^2+\vec{r}_{24}{}^2\right)}{2\vec{r}_{01}{}^2\vec{r}_{04}{}^2\vec{r}_{14}{}^2\vec{r}_{24}{}^2}\right)\ln\left(\frac{\vec{r}_{01}{}^2}{\vec{r}_{14}{}^2}\right). \quad (2.17)$$

$$G_7=G_5|_{1\leftrightarrow 2}, \quad G_8=G_6|_{1\leftrightarrow 2, 0\leftrightarrow 4}. \quad (2.18)$$

After the convolution with $\varepsilon^{i_1j_1h}\varepsilon_{i_3j_3h'}(U_3)^{h'}_h$, (2.11) gives the contribution of the 2-gluon states to the evolution of the 3QWL operator $U_1\cdot U_2\cdot U_3$ describing the total interaction of Wilson lines 1 and 2, leaving Wilson line 3 intact.

$$\begin{aligned}&\mathbf{G}_{12}\varepsilon^{i_1j_1h}\varepsilon_{i_3j_3h'}(U_3)^{h'}_h\\
&=G_1\left(\left(U_0U_4{}^\dagger U_2\right)\cdot\left(U_1U_0{}^\dagger U_4\right)+\left(U_2U_4{}^\dagger U_0\right)\cdot\left(U_4U_0{}^\dagger U_1\right)\right)\cdot U_3\\
&+\left[G_2\left(U_0U_4{}^\dagger U_2U_0{}^\dagger U_4+U_4U_0{}^\dagger U_2U_4{}^\dagger U_0\right)\cdot U_1\cdot U_3+(1\leftrightarrow 2)\right]\\
&-G_4tr\left(U_0U_4{}^\dagger\right)\left(U_1U_0{}^\dagger U_2+U_2U_0{}^\dagger U_1\right)\cdot U_3\cdot U_4\\
&+\left[G_5\ U_2\cdot U_3\cdot U_4\ tr\left(U_0{}^\dagger U_1\right)tr\left(U_0U_4{}^\dagger\right)+(1\leftrightarrow 2)\right]\\
&+\left[G_6\left(tr\left(U_0{}^\dagger U_1\right)\left(U_0U_4{}^\dagger U_2+U_2U_4{}^\dagger U_0\right)\cdot U_3\cdot U_4\right.\right.\\
&\left.\left.+\left(U_2U_4{}^\dagger U_1U_0{}^\dagger U_4+U_4U_0{}^\dagger U_1U_4{}^\dagger U_2\right)\cdot U_0\cdot U_3\right)+(1\leftrightarrow 2, 0\leftrightarrow 4)\right]. \quad (2.19)\end{aligned}$$

One can also write

$$\langle K_{NLO}\otimes(U_1^\dagger)^{j_3}_{j_1}\rangle|_{2g}=-\frac{\alpha_s^2}{8\pi^4}\int d\vec{r}_0d\vec{r}_4\ \mathbf{G}_{1^\dagger}, \quad (2.20)$$

$$\begin{aligned}\mathbf{G}_{1^\dagger}&=G_3\left(U_4{}^\dagger U_0U_1{}^\dagger U_4U_0{}^\dagger+U_0{}^\dagger U_4U_1{}^\dagger U_0U_4{}^\dagger\right.\\
&\left.-tr\left(U_1{}^\dagger U_4\right)tr\left(U_4{}^\dagger U_0\right)U_0{}^\dagger-tr\left(U_0{}^\dagger U_4\right)tr\left(U_1{}^\dagger U_0\right)U_4{}^\dagger\right)^{j_3}_{j_1}\\
&+G_9\left(tr\left(U_1{}^\dagger U_4\right)tr\left(U_4{}^\dagger U_0\right)U_0{}^\dagger-tr\left(U_0{}^\dagger U_4\right)tr\left(U_1{}^\dagger U_0\right)U_4{}^\dagger\right)^{j_3}_{j_1}, \quad (2.21)\end{aligned}$$

$$G_9=\frac{\vec{r}_{01}{}^2-\vec{r}_{04}{}^2+\vec{r}_{14}{}^2}{4\vec{r}_{01}{}^2\vec{r}_{04}{}^2\vec{r}_{14}{}^2}\ln\left(\frac{\vec{r}_{01}{}^2}{\vec{r}_{14}{}^2}\right). \quad (2.22)$$

And then take the convolution

$$\begin{aligned}\mathbf{G}_{\langle 1^\dagger\rangle 3}=\mathbf{G}_{1^\dagger}(U_3)^{j_1}_{j_3}&=G_3\left(tr\left(U_0{}^\dagger U_3U_4{}^\dagger U_0U_1{}^\dagger U_4\right)+tr\left(U_0{}^\dagger U_4U_1{}^\dagger U_0U_4{}^\dagger U_3\right)\right.\\
&\left.-tr\left(U_0{}^\dagger U_3\right)tr\left(U_1{}^\dagger U_4\right)tr\left(U_4{}^\dagger U_0\right)-tr\left(U_0{}^\dagger U_4\right)tr\left(U_1{}^\dagger U_0\right)tr\left(U_4{}^\dagger U_3\right)\right)\\
&+G_9\left(tr\left(U_0{}^\dagger U_3\right)tr\left(U_1{}^\dagger U_4\right)tr\left(U_4{}^\dagger U_0\right)-tr\left(U_0{}^\dagger U_4\right)tr\left(U_1{}^\dagger U_0\right)tr\left(U_4{}^\dagger U_3\right)\right). \quad (2.23)\end{aligned}$$

For the elements of $SU(3)$ group one has the identity

$$\varepsilon^{ijh}\varepsilon_{i'j'h'}(U_1)^{i'}_i(U_1)^{j'}_j=2(U_1^\dagger)^h_{h'}, \quad U_1\cdot U_1\cdot U_3=2tr(U_1^\dagger U_3), \quad (2.24)$$

Taking $\vec{r}_2 = \vec{r}_1$ in (2.19) one can check that it is related to (2.23) via this identity using the other $SU(3)$ identities (A.1) and (A.3). Taking the conjugate of $\mathbf{G}_{1^\dagger}$, one gets

$$\langle K_{NLO} \otimes (U_1)_j^{j'} \rangle|_{2g} = -\frac{\alpha_s^2}{8\pi^4} \int d\vec{r}_0 d\vec{r}_4 \; \mathbf{G}_1, \tag{2.25}$$

$$\mathbf{G}_1 = G_3 \Big(U_4 U_0^\dagger U_1 U_4^\dagger U_0 + U_0 U_4^\dagger U_1 U_0^\dagger U_4$$
$$- tr(U_1 U_4^\dagger) tr(U_4 U_0^\dagger) U_0 - tr(U_0 U_4^\dagger) tr(U_1 U_0^\dagger) U_4 \Big)_j^{j'}$$
$$+ G_9 \Big(tr(U_1 U_4^\dagger) tr(U_4 U_0^\dagger) U_0 - tr(U_0 U_4^\dagger) tr(U_1 U_0^\dagger) U_4 \Big)_j^{j'}, \tag{2.26}$$

The contribution of the evolution of only one line U_1 to the evolution of the 3QWL reads

$$\mathbf{G}_{\langle 1 \rangle 23} = \mathbf{G}_1 \varepsilon^{ijh} \varepsilon_{i'j'h'} (U_2)_i^{i'} (U_3)_h^{h'} = G_3 \Big(U_4 {U_0}^\dagger U_1 {U_4}^\dagger U_0 + U_0 U_4^\dagger U_1 U_0^\dagger U_4$$
$$- tr(U_1 U_4^\dagger) tr(U_4 U_0^\dagger) U_0 - tr\left({U_0}^\dagger U_1\right) tr\left({U_4}^\dagger U_0\right) U_4 \Big) \cdot U_2 \cdot U_3$$
$$+ G_9 \Big(tr(U_1 U_4^\dagger) tr(U_4 U_0^\dagger) U_0 - tr\left({U_0}^\dagger U_1\right) tr\left({U_4}^\dagger U_0\right) U_4 \Big) \cdot U_2 \cdot U_3. \tag{2.27}$$

Then the connected contribution of the evolution of lines 1 and 2 reads

$$\mathbf{G}_{\langle 12 \rangle 3} = \frac{1}{2} [H_1 - (1 \leftrightarrow 2)]$$
$$\times \Big[\left(U_0 {U_4}^\dagger U_2\right) \cdot \left(U_1 {U_0}^\dagger U_4\right) \cdot U_3 - \left(U_0 {U_4}^\dagger U_1\right) \cdot \left(U_2 {U_0}^\dagger U_4\right) \cdot U_3 - (4 \leftrightarrow 0) \Big]$$
$$+ H_2 \Big[tr\left({U_0}^\dagger U_1\right) \left(U_0 {U_4}^\dagger U_2 + U_2 {U_4}^\dagger U_0\right) \cdot U_3 \cdot U_4$$
$$- \left(U_2 {U_0}^\dagger U_1 {U_4}^\dagger U_0 + U_0 {U_4}^\dagger U_1 {U_0}^\dagger U_2\right) \cdot U_3 \cdot U_4 - (4 \leftrightarrow 0) \Big]$$
$$+ H_3 \Big[tr\left({U_0}^\dagger U_1\right) \left(U_0 {U_4}^\dagger U_2 + U_2 {U_4}^\dagger U_0\right) \cdot U_3 \cdot U_4$$
$$+ \left(U_2 {U_0}^\dagger U_1 {U_4}^\dagger U_0 + U_0 {U_4}^\dagger U_1 {U_0}^\dagger U_2\right) \cdot U_3 \cdot U_4 + (4 \leftrightarrow 0) \Big]$$
$$+ H_4 [tr\left(U_0 {U_4}^\dagger\right) \left(U_1 {U_0}^\dagger U_2 + U_2 {U_0}^\dagger U_1\right) \cdot U_3 \cdot U_4$$
$$+ \left(U_0 {U_4}^\dagger U_1\right) \cdot \left(U_2 {U_0}^\dagger U_4\right) \cdot U_3 + \left(U_0 {U_4}^\dagger U_2\right) \cdot \left(U_1 {U_0}^\dagger U_4\right) \cdot U_3 + (4 \leftrightarrow 0)]$$
$$+ H_1 [tr\left(U_4 {U_0}^\dagger\right) \left(U_1 {U_4}^\dagger U_2 + U_2 {U_4}^\dagger U_1\right) \cdot U_0 \cdot U_3 - (4 \leftrightarrow 0)] + (1 \leftrightarrow 2). \tag{2.28}$$

$$H_1 = \frac{1}{8} \Bigg[\frac{\left({\vec{r}_{02}}^2 - {\vec{r}_{12}}^2\right) \left({\vec{r}_{14}}^2 \left({\vec{r}_{02}}^2 - {\vec{r}_{24}}^2\right) + {\vec{r}_{04}}^2 \left({\vec{r}_{24}}^2 - {\vec{r}_{12}}^2\right)\right)}{{\vec{r}_{01}}^2 {\vec{r}_{02}}^2 {\vec{r}_{04}}^2 {\vec{r}_{14}}^2 {\vec{r}_{24}}^2}$$
$$+ \frac{{\vec{r}_{12}}^2 - {\vec{r}_{14}}^2 - {\vec{r}_{24}}^2}{{\vec{r}_{04}}^2 {\vec{r}_{14}}^2 {\vec{r}_{24}}^2} + \frac{{\vec{r}_{24}}^2 - {\vec{r}_{12}}^2 - {\vec{r}_{14}}^2}{{\vec{r}_{02}}^2 {\vec{r}_{04}}^2 {\vec{r}_{14}}^2} \Bigg] \ln \left(\frac{{\vec{r}_{01}}^2}{{\vec{r}_{14}}^2} \right). \tag{2.29}$$

$$H_2 = \frac{1}{8} \Bigg[\frac{{\vec{r}_{12}}^2}{{\vec{r}_{01}}^2 {\vec{r}_{14}}^2 {\vec{r}_{02}}^2} + \frac{{\vec{r}_{12}}^2}{{\vec{r}_{01}}^2 {\vec{r}_{14}}^2 {\vec{r}_{24}}^2} + \frac{{\vec{r}_{01}}^2 - {\vec{r}_{02}}^2}{{\vec{r}_{01}}^2 {\vec{r}_{04}}^2 {\vec{r}_{24}}^2} + 2 \frac{{\vec{r}_{14}}^2 - {\vec{r}_{04}}^2 + {\vec{r}_{01}}^2}{{\vec{r}_{01}}^2 {\vec{r}_{04}}^2 {\vec{r}_{14}}^2}$$

$$-\frac{\vec{r}_{12}{}^2}{\vec{r}_{04}{}^2\vec{r}_{14}{}^2\vec{r}_{24}{}^2}+\frac{\vec{r}_{01}{}^2-\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{04}{}^2\vec{r}_{02}{}^2}-\frac{\vec{r}_{24}{}^2}{\vec{r}_{04}{}^2\vec{r}_{14}{}^2\vec{r}_{02}{}^2}\Bigg]\ln\left(\frac{\vec{r}_{01}{}^2}{\vec{r}_{14}{}^2}\right). \tag{2.30}$$

$$H_3=\frac{1}{8}\Bigg[\frac{\vec{r}_{01}{}^2-\vec{r}_{02}{}^2}{\vec{r}_{01}{}^2\vec{r}_{04}{}^2\vec{r}_{24}{}^2}+\frac{\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{14}{}^2\vec{r}_{24}{}^2}-\frac{\vec{r}_{12}{}^2}{\vec{r}_{04}{}^2\vec{r}_{14}{}^2\vec{r}_{24}{}^2}$$
$$+\frac{\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{04}{}^2\vec{r}_{02}{}^2}+\frac{\vec{r}_{24}{}^2-\vec{r}_{14}{}^2}{\vec{r}_{04}{}^2\vec{r}_{14}{}^2\vec{r}_{02}{}^2}-\frac{\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{14}{}^2\vec{r}_{02}{}^2}\Bigg]\ln\left(\frac{\vec{r}_{01}{}^2}{\vec{r}_{14}{}^2}\right). \tag{2.31}$$

$$H_4=\frac{-1}{4\vec{r}_{04}{}^4}-\frac{1}{8}\Bigg[\frac{\vec{r}_{12}{}^2(\vec{r}_{14}{}^2-\vec{r}_{01}{}^2)(\vec{r}_{02}{}^2+\vec{r}_{24}{}^2)}{\vec{r}_{01}{}^2\vec{r}_{02}{}^2\vec{r}_{04}{}^2\vec{r}_{14}{}^2\vec{r}_{24}{}^2}+\frac{\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{14}{}^2\vec{r}_{24}{}^2}-\frac{\vec{r}_{12}{}^4}{\vec{r}_{01}{}^2\vec{r}_{02}{}^2\vec{r}_{14}{}^2\vec{r}_{24}{}^2}$$
$$+\frac{\vec{r}_{24}{}^2+\vec{r}_{02}{}^2-\vec{r}_{14}{}^2}{\vec{r}_{02}{}^2\vec{r}_{04}{}^2\vec{r}_{14}{}^2}+\frac{\vec{r}_{01}{}^2-\vec{r}_{02}{}^2-\vec{r}_{24}{}^2}{\vec{r}_{01}{}^2\vec{r}_{04}{}^2\vec{r}_{24}{}^2}$$
$$+\frac{1}{\vec{r}_{01}{}^2-\vec{r}_{14}{}^2}\left(\frac{\vec{r}_{12}{}^2-\vec{r}_{02}{}^2}{\vec{r}_{01}{}^2\vec{r}_{02}{}^2}-\frac{\vec{r}_{12}{}^2+\vec{r}_{02}{}^2}{\vec{r}_{02}{}^2\vec{r}_{14}{}^2}-\frac{4\vec{r}_{14}{}^2}{\vec{r}_{04}{}^4}+\frac{8}{\vec{r}_{04}{}^2}\right)$$
$$+\frac{1}{\vec{r}_{01}{}^2\vec{r}_{24}{}^2-\vec{r}_{02}{}^2\vec{r}_{14}{}^2}\left(\frac{2\vec{r}_{12}{}^4}{\vec{r}_{02}{}^2\vec{r}_{14}{}^2}+\frac{4\vec{r}_{02}{}^2\vec{r}_{14}{}^2}{\vec{r}_{04}{}^4}-\frac{8\vec{r}_{12}{}^2}{\vec{r}_{04}{}^2}\right)\Bigg]\ln\left(\frac{\vec{r}_{01}{}^2}{\vec{r}_{14}{}^2}\right). \tag{2.32}$$

The fully connected contribution can be taken from [2] or [11] and transformed to the form

$$\mathbf{G}_{\langle 123\rangle}=H_5\Big[\Big(U_0U_4{}^\dagger U_3\Big)\cdot\Big(U_1U_0{}^\dagger U_2\Big)\cdot U_4-\Big(U_0U_4{}^\dagger U_2\Big)\cdot\Big(U_1U_0{}^\dagger U_3\Big)\cdot U_4$$
$$+\Big(U_2U_0{}^\dagger U_1\Big)\cdot\Big(U_3U_4{}^\dagger U_0\Big)\cdot U_4-\Big(U_2U_0{}^\dagger U_4\Big)\cdot\Big(U_3U_4{}^\dagger U_1\Big)\cdot U_0+(4\leftrightarrow 0)\Big]$$
$$+H_6[\Big(U_0U_4{}^\dagger U_2\Big)\cdot\Big(U_1U_0{}^\dagger U_3\Big)\cdot U_4+\Big(U_0U_4{}^\dagger U_3\Big)\cdot\Big(U_1U_0{}^\dagger U_2\Big)\cdot U_4$$
$$+\Big(U_2U_0{}^\dagger U_1\Big)\cdot\Big(U_3U_4{}^\dagger U_0\Big)\cdot U_4-\Big(U_2U_0{}^\dagger U_4\Big)\cdot\Big(U_3U_4{}^\dagger U_1\Big)\cdot U_0-(4\leftrightarrow 0)]$$
$$+(1\leftrightarrow 2)+(1\leftrightarrow 3). \tag{2.33}$$

$$H_5=\frac{1}{8}\Bigg[\frac{\vec{r}_{13}{}^2\vec{r}_{02}{}^2}{\vec{r}_{01}{}^2\vec{r}_{03}{}^2\vec{r}_{04}{}^2\vec{r}_{24}{}^2}-\frac{\vec{r}_{12}{}^2\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{03}{}^2\vec{r}_{14}{}^2\vec{r}_{24}{}^2}+\frac{\vec{r}_{12}{}^2\vec{r}_{34}{}^2}{\vec{r}_{03}{}^2\vec{r}_{04}{}^2\vec{r}_{14}{}^2\vec{r}_{24}{}^2}-\frac{\vec{r}_{13}{}^2\vec{r}_{24}{}^2}{\vec{r}_{04}{}^2\vec{r}_{14}{}^2\vec{r}_{34}{}^2\vec{r}_{02}{}^2}$$
$$+\frac{\vec{r}_{12}{}^2\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{14}{}^2\vec{r}_{34}{}^2\vec{r}_{02}{}^2}-\frac{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{04}{}^2\vec{r}_{34}{}^2\vec{r}_{02}{}^2}+\frac{\vec{r}_{14}{}^2-\vec{r}_{34}{}^2}{\vec{r}_{03}{}^2\vec{r}_{04}{}^2\vec{r}_{14}{}^2}-\frac{\vec{r}_{12}{}^2}{\vec{r}_{04}{}^2\vec{r}_{14}{}^2\vec{r}_{24}{}^2}+\frac{\vec{r}_{03}{}^2-\vec{r}_{01}{}^2}{\vec{r}_{01}{}^2\vec{r}_{04}{}^2\vec{r}_{34}{}^2}$$
$$-\frac{\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{14}{}^2\vec{r}_{34}{}^2}+\frac{\vec{r}_{13}{}^2}{\vec{r}_{04}{}^2\vec{r}_{14}{}^2\vec{r}_{34}{}^2}+\frac{\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{04}{}^2\vec{r}_{02}{}^2}+\frac{\vec{r}_{24}{}^2}{\vec{r}_{04}{}^2\vec{r}_{14}{}^2\vec{r}_{02}{}^2}+\frac{\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{03}{}^2\vec{r}_{14}{}^2}+\frac{\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{14}{}^2\vec{r}_{24}{}^2}$$
$$+\frac{\vec{r}_{03}{}^2-\vec{r}_{23}{}^2}{\vec{r}_{03}{}^2\vec{r}_{04}{}^2\vec{r}_{24}{}^2}-\frac{\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{14}{}^2\vec{r}_{02}{}^2}+\frac{\vec{r}_{23}{}^2-\vec{r}_{34}{}^2}{\vec{r}_{04}{}^2\vec{r}_{34}{}^2\vec{r}_{02}{}^2}-\frac{\vec{r}_{02}{}^2}{\vec{r}_{01}{}^2\vec{r}_{04}{}^2\vec{r}_{24}{}^2}-\frac{\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{03}{}^2\vec{r}_{04}{}^2}\Bigg]\ln\left(\frac{\vec{r}_{01}{}^2}{\vec{r}_{14}{}^2}\right). \tag{2.34}$$

$$H_6=\frac{1}{8}\Bigg[\frac{\vec{r}_{12}{}^2\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{03}{}^2\vec{r}_{14}{}^2\vec{r}_{24}{}^2}-\frac{\vec{r}_{13}{}^2\vec{r}_{02}{}^2}{\vec{r}_{01}{}^2\vec{r}_{03}{}^2\vec{r}_{04}{}^2\vec{r}_{24}{}^2}-\frac{\vec{r}_{12}{}^2\vec{r}_{34}{}^2}{\vec{r}_{03}{}^2\vec{r}_{04}{}^2\vec{r}_{14}{}^2\vec{r}_{24}{}^2}$$
$$-\frac{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{04}{}^2\vec{r}_{34}{}^2\vec{r}_{02}{}^2}-\frac{\vec{r}_{13}{}^2\vec{r}_{24}{}^2}{\vec{r}_{04}{}^2\vec{r}_{14}{}^2\vec{r}_{34}{}^2\vec{r}_{02}{}^2}+\frac{\vec{r}_{12}{}^2\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{14}{}^2\vec{r}_{34}{}^2\vec{r}_{02}{}^2}+\frac{\vec{r}_{03}{}^2-\vec{r}_{01}{}^2}{\vec{r}_{01}{}^2\vec{r}_{04}{}^2\vec{r}_{34}{}^2}+\frac{\vec{r}_{13}{}^2-\vec{r}_{01}{}^2}{\vec{r}_{01}{}^2\vec{r}_{03}{}^2\vec{r}_{04}{}^2}$$
$$+\frac{\vec{r}_{23}{}^2-\vec{r}_{03}{}^2}{\vec{r}_{03}{}^2\vec{r}_{04}{}^2\vec{r}_{24}{}^2}+2\frac{\vec{r}_{04}{}^2-\vec{r}_{14}{}^2-\vec{r}_{01}{}^2}{\vec{r}_{01}{}^2\vec{r}_{14}{}^2\vec{r}_{04}{}^2}+\frac{\vec{r}_{24}{}^2-\vec{r}_{14}{}^2}{\vec{r}_{04}{}^2\vec{r}_{14}{}^2\vec{r}_{02}{}^2}+\frac{\vec{r}_{02}{}^2}{\vec{r}_{01}{}^2\vec{r}_{04}{}^2\vec{r}_{24}{}^2}+\frac{\vec{r}_{34}{}^2}{\vec{r}_{03}{}^2\vec{r}_{04}{}^2\vec{r}_{14}{}^2}$$
$$+\frac{\vec{r}_{12}{}^2}{\vec{r}_{04}{}^2\vec{r}_{14}{}^2\vec{r}_{24}{}^2}-\frac{\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{14}{}^2\vec{r}_{34}{}^2}+\frac{\vec{r}_{13}{}^2}{\vec{r}_{04}{}^2\vec{r}_{14}{}^2\vec{r}_{34}{}^2}-\frac{\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{14}{}^2\vec{r}_{24}{}^2}-\frac{\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{03}{}^2\vec{r}_{14}{}^2}$$
$$+\frac{\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{04}{}^2\vec{r}_{02}{}^2}-\frac{\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{14}{}^2\vec{r}_{02}{}^2}+\frac{\vec{r}_{23}{}^2}{\vec{r}_{04}{}^2\vec{r}_{34}{}^2\vec{r}_{02}{}^2}\Bigg]\ln\left(\frac{\vec{r}_{01}{}^2}{\vec{r}_{14}{}^2}\right). \tag{2.35}$$

3 Construction of the kernel

Taking the contributions of the self-interaction on one Wilson line (2.27), the connected contributions of 2 (2.28) and 3 (2.33) Wilson lines from the previous section one can write for the full contribution to the evolution of the 3QWL with 2-gluons intersecting the shockwave

$$\langle K_{NLO} \otimes B_{123} \rangle|_{2g} = \langle K_{NLO} \otimes U_1 \cdot U_2 \cdot U_3 \rangle|_{2g} = -\frac{\alpha_s^2}{8\pi^4} \int d\vec{r}_0 d\vec{r}_4 \; \mathbf{G}, \tag{3.1}$$

$$\mathbf{G} = \mathbf{G}_{\langle 1 \rangle 23} + \mathbf{G}_{1 \langle 2 \rangle 3} + \mathbf{G}_{12 \langle 3 \rangle} + \mathbf{G}_{\langle 12 \rangle 3} + \mathbf{G}_{1 \langle 23 \rangle} + \mathbf{G}_{\langle 13 \rangle 2} + \mathbf{G}_{\langle 123 \rangle}, \tag{3.2}$$

Here $\langle \dots \rangle$ stands for the connected contribution, i.e. $\mathbf{G}_{\langle 1 \rangle 23}$ gives the contribution of the evolution of line 1 (2.27), with lines 2 and 3 being spectators, $\mathbf{G}_{\langle 12 \rangle 3}$ — the connected contribution of the evolution of lines 1 and 2 (2.28), with line 3 being intact, and $\mathbf{G}_{\langle 123 \rangle}$ — the fully connected contribution (2.33). All the rest can be obtained from them by $1 \leftrightarrow 2 \leftrightarrow 3$ transformation.

There are several useful $SU(3)$ identities, which help to reduce the number of color structures. They are listed in the appendix A. First we use (A.5) to get rid of the structure

$$\left(U_0 U_4{}^\dagger U_3 U_0{}^\dagger U_4\right) \cdot U_1 \cdot U_2 \tag{3.3}$$

and the 2 ones it goes into after the $1 \leftrightarrow 2 \leftrightarrow 3$ transformations with their symmetric counterparts w.r.t. $0 \leftrightarrow 4$ exchange. Next we use (A.6) to eliminate 6 such contributions antisymmetric w.r.t. $0 \leftrightarrow 4$ exchange as

$$\left(U_2 U_0{}^\dagger U_1 U_4{}^\dagger U_0 + U_0 U_4{}^\dagger U_1 U_0{}^\dagger U_2\right) \cdot U_3 \cdot U_4 - (4 \leftrightarrow 0). \tag{3.4}$$

After that we use (A.7) to express 6 structures like

$$\left(U_2 U_4{}^\dagger U_1 U_0{}^\dagger U_4 + U_4 U_0{}^\dagger U_1 U_4{}^\dagger U_2\right) \cdot U_0 \cdot U_3 \tag{3.5}$$

and their symmetric counterparts w.r.t. $0 \leftrightarrow 4$ exchange through other structures. Then via (A.8) we cancel 3 structures of the form

$$U_2 \cdot U_3 \cdot U_4 \; tr\left(U_0{}^\dagger U_1\right) tr\left(U_0 U_4{}^\dagger\right) - U_2 \cdot U_3 \cdot U_0 \; tr\left(U_4{}^\dagger U_1\right) tr\left(U_4 U_0{}^\dagger\right). \tag{3.6}$$

Finally, by means of (A.9) we discard the 3 nonconformal terms proportional to

$$tr\left(U_0 U_4{}^\dagger\right)\left(U_1 U_0{}^\dagger U_2 + U_2 U_0{}^\dagger U_1\right) \cdot U_3 \cdot U_4 - (4 \leftrightarrow 0) \tag{3.7}$$

and the 2 structures they go into after the $1 \leftrightarrow 2 \leftrightarrow 3$ transformations. Finally, we get

$$\mathbf{G} = \{(L_{12} + \tilde{L}_{12})\left(U_0 U_4{}^\dagger U_2\right) \cdot \left(U_1 U_0{}^\dagger U_4\right) \cdot U_3 + L_{12} tr\left(U_0 U_4{}^\dagger\right)\left(U_1 U_0{}^\dagger U_2\right) \cdot U_3 \cdot U_4$$
$$+ (M_{13} - M_{12} - M_{23} + M_2)\left[\left(U_0 U_4{}^\dagger U_3\right) \cdot \left(U_2 U_0{}^\dagger U_1\right) \cdot U_4 + \left(U_1 U_0{}^\dagger U_2\right) \cdot \left(U_3 U_4{}^\dagger U_0\right) \cdot U_4\right]$$
$$+ (\text{all 5 permutations}\, 1 \leftrightarrow 2 \leftrightarrow 3)\} + (0 \leftrightarrow 4). \tag{3.8}$$

$$L_{12}=\left[\frac{1}{\vec{r}_{01}{}^2\vec{r}_{24}{}^2-\vec{r}_{02}{}^2\vec{r}_{14}{}^2}\left(-\frac{\vec{r}_{12}{}^4}{8}\left(\frac{1}{\vec{r}_{01}{}^2\vec{r}_{24}{}^2}+\frac{1}{\vec{r}_{02}{}^2\vec{r}_{14}{}^2}\right)+\frac{\vec{r}_{12}{}^2}{\vec{r}_{04}{}^2}-\frac{\vec{r}_{02}{}^2\vec{r}_{14}{}^2+\vec{r}_{01}{}^2\vec{r}_{24}{}^2}{4\vec{r}_{04}{}^4}\right)\right.$$
$$\left.+\frac{\vec{r}_{12}{}^2}{8\vec{r}_{04}{}^2}\left(\frac{1}{\vec{r}_{02}{}^2\vec{r}_{14}{}^2}-\frac{1}{\vec{r}_{01}{}^2\vec{r}_{24}{}^2}\right)\right]\ln\left(\frac{\vec{r}_{01}{}^2\vec{r}_{24}{}^2}{\vec{r}_{14}{}^2\vec{r}_{02}{}^2}\right)+\frac{1}{2\vec{r}_{04}{}^4}. \tag{3.9}$$

$$\tilde{L}_{12}=\frac{\vec{r}_{12}{}^2}{8}\left[\frac{\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{02}{}^2\vec{r}_{14}{}^2\vec{r}_{24}{}^2}-\frac{1}{\vec{r}_{01}{}^2\vec{r}_{04}{}^2\vec{r}_{24}{}^2}-\frac{1}{\vec{r}_{02}{}^2\vec{r}_{04}{}^2\vec{r}_{14}{}^2}\right]\ln\left(\frac{\vec{r}_{01}{}^2\vec{r}_{24}{}^2}{\vec{r}_{14}{}^2\vec{r}_{02}{}^2}\right). \tag{3.10}$$

$$M_{12}=\frac{\vec{r}_{12}{}^2}{16}\left[\frac{\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{02}{}^2\vec{r}_{14}{}^2\vec{r}_{24}{}^2}-\frac{1}{\vec{r}_{01}{}^2\vec{r}_{04}{}^2\vec{r}_{24}{}^2}-\frac{1}{\vec{r}_{02}{}^2\vec{r}_{04}{}^2\vec{r}_{14}{}^2}\right]\ln\left(\frac{\vec{r}_{01}{}^2\vec{r}_{02}{}^2}{\vec{r}_{14}{}^2\vec{r}_{24}{}^2}\right). \tag{3.11}$$

$$M_2=\left(\frac{\vec{r}_{12}{}^2\vec{r}_{23}{}^2}{\vec{r}_{01}{}^2\vec{r}_{02}{}^2\vec{r}_{24}{}^2\vec{r}_{34}{}^2}-\frac{\vec{r}_{14}{}^2\vec{r}_{23}{}^2}{\vec{r}_{01}{}^2\vec{r}_{04}{}^2\vec{r}_{24}{}^2\vec{r}_{34}{}^2}-\frac{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{02}{}^2\vec{r}_{04}{}^2\vec{r}_{34}{}^2}+\frac{\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{04}{}^2\vec{r}_{34}{}^2}\right)$$
$$\times\frac{1}{4}\ln\left(\frac{\vec{r}_{02}{}^2}{\vec{r}_{24}{}^2}\right). \tag{3.12}$$

These functions obey the identities

$$M_2|_{\vec{r}_1\to\vec{r}_3}=\frac{\vec{r}_{23}{}^2}{4}\left(\frac{\vec{r}_{23}{}^2}{\vec{r}_{03}{}^2\vec{r}_{02}{}^2\vec{r}_{24}{}^2\vec{r}_{34}{}^2}-\frac{1}{\vec{r}_{03}{}^2\vec{r}_{04}{}^2\vec{r}_{24}{}^2}-\frac{1}{\vec{r}_{02}{}^2\vec{r}_{04}{}^2\vec{r}_{34}{}^2}\right)\ln\left(\frac{\vec{r}_{02}{}^2}{\vec{r}_{24}{}^2}\right). \tag{3.13}$$

$$M_{13}-M_{12}-M_{23}+M_2|_{\vec{r}_1\to\vec{r}_3}=\tilde{L}_{23}. \tag{3.14}$$

$$M_{13}-M_{12}-M_{23}+M_2|_{\vec{r}_1\to\vec{r}_2}=M_{13}-M_{12}-M_{23}+M_2|_{\vec{r}_3\to\vec{r}_2}=0. \tag{3.15}$$

Using these identities and (A.1) with $l=3$, we get the dipole result

$$\mathbf{G}|_{\vec{r}_1\to\vec{r}_3}=4(L_{32}+\tilde{L}_{32})tr\left(U_0{}^\dagger U_4\right)tr\left(U_3{}^\dagger U_0\right)tr\left(U_4{}^\dagger U_2\right)$$
$$-4L_{32}tr\left(U_0{}^\dagger U_2U_4{}^\dagger U_0U_3{}^\dagger U_4\right)+(0\leftrightarrow 4). \tag{3.16}$$

This expression is twice the corresponding part of the BK kernel for $tr(U_2U_3^\dagger)$.

The only UV divergent term in (3.8) is the term proportional to L_{12}. This term has the same coordinate structure as the corresponding term in the dipole kernel. Therefore we can do the same subtraction as in the dipole case. Using (A.3), we get

$$\left(U_0U_4{}^\dagger U_2\right)\cdot\left(U_1U_0{}^\dagger U_4\right)\cdot U_3+tr\left(U_0U_4{}^\dagger\right)\left(U_1U_0{}^\dagger U_2\right)\cdot U_3\cdot U_4+(2\leftrightarrow 1)|_{\vec{r}_0\to\vec{r}_4}$$
$$=3[tr\left(U_1U_4{}^\dagger\right)U_2\cdot U_3\cdot U_4+tr\left(U_2U_4{}^\dagger\right)U_1\cdot U_3\cdot U_4-tr\left(U_3U_4{}^\dagger\right)U_1\cdot U_2\cdot U_4]-U_1\cdot U_2\cdot U_3$$
$$=\frac{3}{2}[B_{144}B_{234}+B_{244}B_{134}-B_{344}B_{124}]-B_{123}, \tag{3.17}$$

$$B_{123}=U_1\cdot U_2\cdot U_3=\varepsilon^{i'j'h'}\varepsilon_{ijh}U^i_{1i'}U^j_{2j'}U^h_{3h'}. \tag{3.18}$$

Therefore we can separate the result into the UV finite and divergent parts

$$\langle K_{NLO}\otimes B_{123}\rangle|_{2g}=-\frac{\alpha_s^2}{8\pi^4}\int d\vec{r}_0d\vec{r}_4\ \mathbf{G}_{finite}-\frac{\alpha_s^2}{8\pi^3}\int d\vec{r}_0\ \mathbf{G}_{UV}, \tag{3.19}$$

$$\mathbf{G}_{finite}=\{\tilde{L}_{12}\left(U_0U_4{}^\dagger U_2\right)\cdot\left(U_1U_0{}^\dagger U_4\right)\cdot U_3$$

$$+L_{12}\left[\left(U_0{U_4}^\dagger U_2\right)\cdot\left(U_1{U_0}^\dagger U_4\right)\cdot U_3+tr\left(U_0{U_4}^\dagger\right)\left(U_1{U_0}^\dagger U_2\right)\cdot U_3\cdot U_4\right.$$

$$\left.-\frac{3}{4}[B_{144}B_{234}+B_{244}B_{134}-B_{344}B_{124}]+\frac{1}{2}B_{123}\right]$$

$$+(M_{13}-M_{12}-M_{23}+M_2)\left[\left(U_0{U_4}^\dagger U_3\right)\cdot\left(U_2{U_0}^\dagger U_1\right)\cdot U_4+\left(U_1{U_0}^\dagger U_2\right)\cdot\left(U_3{U_4}^\dagger U_0\right)\cdot U_4\right]$$

$$+\,(\text{all 5 permutations}\,1\leftrightarrow 2\leftrightarrow 3)\}+(0\leftrightarrow 4). \tag{3.20}$$

And $\mathbf{G}_{UV}$ is included into the term describing the contribution with one gluon crossing the shockwave in [2].

The contribution of the diagrams with 1 gluon intersecting the shockwave, which are not proportional to the β-function one can take from (5.27) in [11]

$$\langle\tilde{K}_{NLO}\otimes B_{123}\rangle|_{1g}=\frac{\alpha_s^2}{(2\pi)^3}\int d\vec{r}_0\left[\frac{(\vec{r}_{10}\vec{r}_{20})}{\vec{r}_{10}^{\,2}\vec{r}_{20}^{\,2}}-\frac{(\vec{r}_{30}\vec{r}_{20})}{\vec{r}_{30}^{\,2}\vec{r}_{20}^{\,2}}\right]\ln\frac{\vec{r}_{30}^{\,2}}{\vec{r}_{31}^{\,2}}\ln\frac{\vec{r}_{10}^{\,2}}{\vec{r}_{31}^{\,2}}\left(B_{100}B_{320}-B_{300}B_{210}\right)$$

$$+\frac{\alpha_s^2}{(2\pi)^3}\int d\vec{r}_0\left[\frac{1}{\vec{r}_{10}^{\,2}}-\frac{(\vec{r}_{30}\vec{r}_{10})}{\vec{r}_{30}^{\,2}\vec{r}_{10}^{\,2}}\right]\ln\frac{\vec{r}_{30}^{\,2}}{\vec{r}_{31}^{\,2}}\ln\frac{\vec{r}_{10}^{\,2}}{\vec{r}_{31}^{\,2}}\left(B_{123}-\frac{1}{2}\left[3B_{100}B_{320}+B_{300}B_{120}-B_{200}B_{130}\right]\right)$$

$$+\frac{\alpha_s^2}{(2\pi)^3}\int d\vec{r}_0\left[\frac{(\vec{r}_{10}\vec{r}_{30})}{\vec{r}_{10}^{\,2}\vec{r}_{30}^{\,2}}-\frac{1}{\vec{r}_{30}^{\,2}}\right]\ln\frac{\vec{r}_{30}^{\,2}}{\vec{r}_{31}^{\,2}}\ln\frac{\vec{r}_{10}^{\,2}}{\vec{r}_{31}^{\,2}}\left(\frac{1}{2}\left[3B_{300}B_{120}+B_{100}B_{320}-B_{200}B_{130}\right]-B_{123}\right)$$

$$+\,(2\leftrightarrow 1)+(2\leftrightarrow 3). \tag{3.21}$$

This term has the correct dipole limit (see (5.28) in [11]).

The contribution proportional to β-function reads (from [2])

$$\langle\tilde{K}_{NLO}\otimes B_{123}\rangle|_{1g}^{\beta}=\left[-\frac{\alpha_s^2}{(2\pi)^3}\frac{11}{2}\int d\vec{r}_0\left[\ln\left(\frac{\vec{r}_{01}^{\,2}}{\vec{r}_{02}^{\,2}}\right)\left(\frac{1}{\vec{r}_{02}^{\,2}}-\frac{1}{\vec{r}_{01}^{\,2}}\right)-\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}}\ln\left(\frac{\vec{r}_{12}^{\,2}}{\tilde{\mu}^2}\right)\right.\right.$$

$$\left.+\frac{1}{\vec{r}_{02}^{\,2}}\ln\left(\frac{\vec{r}_{02}^{\,2}}{\tilde{\mu}^2}\right)+\frac{1}{\vec{r}_{01}^{\,2}}\ln\left(\frac{\vec{r}_{01}^{\,2}}{\tilde{\mu}^2}\right)\right]$$

$$\left.\times\left(U_0\cdot U_3\cdot(U_2U_0^\dagger U_1)+U_0\cdot U_3\cdot(U_1U_0^\dagger U_2)+\frac{2}{3}U_1\cdot U_2\cdot U_3\right)+(1\leftrightarrow 3)+(2\leftrightarrow 3)\right]$$

$$+\left[\frac{\alpha_s^2}{(2\pi)^3}11\int\frac{d\vec{r}_0}{\vec{r}_{01}^{\,2}}\ln\left(\frac{\vec{r}_{01}^{\,2}}{\tilde{\mu}^2}\right)\left(U_0\cdot U_2\cdot U_3tr(U_1U_0^\dagger)-\frac{1}{3}U_1\cdot U_2\cdot U_3\right)+(1\leftrightarrow 3)+(1\leftrightarrow 2)\right], \tag{3.22}$$

$$\frac{11}{3}\ln\frac{1}{\tilde{\mu}^2}=\frac{11}{3}\ln\left(\frac{\mu^2}{4e^{2\psi(1)}}\right)+\frac{67}{9}-\frac{\pi^2}{3}. \tag{3.23}$$

Or, after some algebra

$$\langle\tilde{K}_{NLO}\otimes B_{123}\rangle|_{1g}^{\beta}=-\frac{\alpha_s^2}{(2\pi)^3}\frac{11}{6}\int d\vec{r}_0\left[\ln\left(\frac{\vec{r}_{01}^{\,2}}{\vec{r}_{02}^{\,2}}\right)\left(\frac{1}{\vec{r}_{02}^{\,2}}-\frac{1}{\vec{r}_{01}^{\,2}}\right)-\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}}\ln\left(\frac{\vec{r}_{12}^{\,2}}{\tilde{\mu}^2}\right)\right]$$

$$\times\left(\frac{3}{2}(B_{100}B_{230}+B_{200}B_{130}-B_{300}B_{210})-B_{123}\right)+(1\leftrightarrow 3)+(2\leftrightarrow 3). \tag{3.24}$$

It also has the correct dipole limit

$$\langle \tilde{K}_{NLO} \otimes B_{122} \rangle|^{\beta}_{1g} = -\frac{\alpha_s^2}{(2\pi)^3} \frac{11}{3} \int d\vec{r}_0 \left[\ln\left(\frac{\vec{r}_{01}^{\,2}}{\vec{r}_{02}^{\,2}}\right) \left(\frac{1}{\vec{r}_{02}^{\,2}} - \frac{1}{\vec{r}_{01}^{\,2}}\right) - \frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}} \ln\left(\frac{\vec{r}_{12}^{\,2}}{\tilde{\mu}^2}\right) \right] \times \left(\frac{3}{2} B_{100}B_{220} - B_{122}\right). \tag{3.25}$$

and it matches the BFKL kernel [15]. Therefore the real part of the whole kernel reads

$$\langle K_{NLO} \otimes B_{123} \rangle|_{\text{real}} = -\frac{\alpha_s^2}{8\pi^4} \int d\vec{r}_0 d\vec{r}_4 \, \mathbf{G}_{finite} - \frac{\alpha_s^2}{8\pi^3} \int d\vec{r}_0 \, \mathbf{G}_{\text{real}}, \tag{3.26}$$

$$\begin{aligned}
\mathbf{G}_{\text{real}} = & -\frac{1}{2} \left[\frac{(\vec{r}_{10}\vec{r}_{20})}{\vec{r}_{10}^{\,2}\vec{r}_{20}^{\,2}} - \frac{(\vec{r}_{30}\vec{r}_{20})}{\vec{r}_{30}^{\,2}\vec{r}_{20}^{\,2}} \right] \ln\frac{\vec{r}_{30}^{\,2}}{\vec{r}_{31}^{\,2}} \ln\frac{\vec{r}_{10}^{\,2}}{\vec{r}_{31}^{\,2}} \left(B_{100}B_{320} - B_{300}B_{210}\right) \\
& - \left[\frac{1}{\vec{r}_{10}^{\,2}} - \frac{(\vec{r}_{30}\vec{r}_{10})}{\vec{r}_{30}^{\,2}\vec{r}_{10}^{\,2}} \right] \ln\frac{\vec{r}_{30}^{\,2}}{\vec{r}_{31}^{\,2}} \ln\frac{\vec{r}_{10}^{\,2}}{\vec{r}_{31}^{\,2}} \left(B_{123} - \frac{1}{2}\left[3B_{100}B_{320} + B_{300}B_{120} - B_{200}B_{130}\right] \right) \\
& + \frac{11}{12} \left[\ln\left(\frac{\vec{r}_{01}^{\,2}}{\vec{r}_{02}^{\,2}}\right) \left(\frac{1}{\vec{r}_{02}^{\,2}} - \frac{1}{\vec{r}_{01}^{\,2}}\right) - \frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}} \ln\left(\frac{\vec{r}_{12}^{\,2}}{\tilde{\mu}^2}\right) \right] \\
& \times \left(\frac{3}{2}(B_{100}B_{230} + B_{200}B_{130} - B_{300}B_{210}) - B_{123} \right) \\
& + (\text{all 5 permutations } 1 \leftrightarrow 2 \leftrightarrow 3),
\end{aligned} \tag{3.27}$$

and $\mathbf{G}_{finite}$ is defined in (3.20). If we put $\vec{r}_2 = \vec{r}_3$ here, we get the dipole result (see (100) in [5])

$$\mathbf{G}_{\text{real}}|_{\vec{r}_2=\vec{r}_3} = \left\{ \frac{11}{3} \left[\ln\left(\frac{\vec{r}_{01}^{\,2}}{\vec{r}_{02}^{\,2}}\right) \left(\frac{1}{\vec{r}_{02}^{\,2}} - \frac{1}{\vec{r}_{01}^{\,2}}\right) - \frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}} \ln\left(\frac{\vec{r}_{12}^{\,2}}{\tilde{\mu}^2}\right) \right] + 2\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{20}^{\,2}\vec{r}_{10}^{\,2}} \ln\frac{\vec{r}_{20}^{\,2}}{\vec{r}_{21}^{\,2}} \ln\frac{\vec{r}_{10}^{\,2}}{\vec{r}_{21}^{\,2}} \right\} \times \left(\frac{3}{2} B_{100}B_{220} - B_{122}\right). \tag{3.28}$$

Finally, from the condition that the kernel must vanish without the shockwave (if all the $B = 6$) and that the virtual contribution is proportional to B_{123}, we get the total kernel

$$\langle K_{NLO} \otimes B_{123} \rangle = -\frac{\alpha_s^2}{8\pi^4} \int d\vec{r}_0 d\vec{r}_4 \, \mathbf{G}_{finite} - \frac{\alpha_s^2}{8\pi^3} \int d\vec{r}_0 \, \mathbf{G}', \tag{3.29}$$

$$\begin{aligned}
\mathbf{G}' = & \frac{1}{2} \left[\frac{\vec{r}_{13}^{\,2}}{\vec{r}_{10}^{\,2}\vec{r}_{30}^{\,2}} - \frac{\vec{r}_{32}^{\,2}}{\vec{r}_{30}^{\,2}\vec{r}_{20}^{\,2}} \right] \ln\frac{\vec{r}_{20}^{\,2}}{\vec{r}_{21}^{\,2}} \ln\frac{\vec{r}_{10}^{\,2}}{\vec{r}_{21}^{\,2}} \left(B_{100}B_{320} - B_{200}B_{310}\right) \\
& - \frac{\vec{r}_{12}^{\,2}}{\vec{r}_{10}^{\,2}\vec{r}_{20}^{\,2}} \ln\frac{\vec{r}_{10}^{\,2}}{\vec{r}_{12}^{\,2}} \ln\frac{\vec{r}_{20}^{\,2}}{\vec{r}_{12}^{\,2}} \left(9B_{123} - \frac{1}{2}\left[2\left(B_{100}B_{320} + B_{200}B_{130}\right) - B_{300}B_{120}\right] \right) \\
& + \frac{11}{6} \left[\ln\left(\frac{\vec{r}_{01}^{\,2}}{\vec{r}_{02}^{\,2}}\right) \left(\frac{1}{\vec{r}_{02}^{\,2}} - \frac{1}{\vec{r}_{01}^{\,2}}\right) - \frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}} \ln\left(\frac{\vec{r}_{12}^{\,2}}{\tilde{\mu}^2}\right) \right] \\
& \times \left(\frac{3}{2}(B_{100}B_{230} + B_{200}B_{130} - B_{300}B_{210}) - 9B_{123} \right) + (1 \leftrightarrow 3) + (2 \leftrightarrow 3).
\end{aligned} \tag{3.30}$$

It differs from (3.27) in the coefficients of B_{123}'s which turn into 9's; $\mathbf{G}_{finite}$ is defined in (3.20).

4 Evolution equation for composite 3QWL operator

To construct composite conformal operators we will use the model

$$O^{conf} = O + \frac{1}{2}\frac{\partial O}{\partial\eta}\bigg|_{\frac{\vec r_{mn}^{\,2}}{\vec r_{im}^{\,2}\vec r_{in}^{\,2}} \to \frac{\vec r_{mn}^{\,2}}{\vec r_{im}^{\,2}\vec r_{in}^{\,2}}\ln\left(\frac{\vec r_{mn}^{\,2}a}{\vec r_{im}^{\,2}\vec r_{in}^{\,2}}\right)}, \tag{4.1}$$

where a is an arbitrary constant. For the conformal 3QWL operator we have the following ansatz

$$B_{123}^{conf} = B_{123} + \frac{\alpha_s 3}{8\pi^2}\int d\vec r_4 \left[\frac{\vec r_{12}^{\,2}}{\vec r_{41}^{\,2}\vec r_{42}^{\,2}}\ln\left(\frac{a\vec r_{12}^{\,2}}{\vec r_{41}^{\,2}\vec r_{42}^{\,2}}\right)\right.$$
$$\left.\times(-B_{123} + \frac{1}{6}(B_{144}B_{324} + B_{244}B_{314} - B_{344}B_{214})) + (1 \leftrightarrow 3) + (2 \leftrightarrow 3)\right]. \tag{4.2}$$

If we put $\vec r_2 = \vec r_3$, then

$$B_{122}^{conf} = B_{122} + \frac{\alpha_s 3}{4\pi^2}\int d\vec r_4 \frac{\vec r_{12}^{\,2}}{\vec r_{41}^{\,2}\vec r_{42}^{\,2}}\ln\left(\frac{a\vec r_{12}^{\,2}}{\vec r_{41}^{\,2}\vec r_{42}^{\,2}}\right)(-B_{122} + \frac{1}{6}B_{144}B_{224}),$$

or

$$tr(U_1U_2^\dagger)^{conf} = tr(U_1U_2^\dagger) + \frac{\alpha_s}{4\pi^2}\int d\vec r_4 \frac{\vec r_{12}^{\,2}}{\vec r_{41}^{\,2}\vec r_{42}^{\,2}}\ln\left(\frac{a\vec r_{12}^{\,2}}{\vec r_{41}^{\,2}\vec r_{42}^{\,2}}\right)(tr(U_1U_4^\dagger)tr(U_4U_2^\dagger) - 3tr(U_1U_2^\dagger)), \tag{4.3}$$

which is exactly the composite dipole operator of [6]. Using SU(3) identity (A.3) one can rewrite (4.2) as

$$B_{123}^{conf} = B_{123} + \frac{\alpha_s}{8\pi^2}\int d\vec r_4 \left[\frac{\vec r_{12}^{\,2}}{\vec r_{41}^{\,2}\vec r_{42}^{\,2}}\ln\left(\frac{a\vec r_{12}^{\,2}}{\vec r_{41}^{\,2}\vec r_{42}^{\,2}}\right)(\left(U_2U_4^\dagger U_1 + U_1U_4^\dagger U_2\right)\cdot U_4\cdot U_3 - 2B_{123})\right.$$
$$\left. + (1 \leftrightarrow 3) + (2 \leftrightarrow 3)\right]. \tag{4.4}$$

Then as in [6], for $(-B_{123} + \frac{1}{6}(B_{144}B_{324} + B_{244}B_{314} - B_{344}B_{214}))$ we have

$$(-3B_{123} + \frac{1}{2}(B_{144}B_{324} + B_{244}B_{314} - B_{344}B_{214}))^{conf}$$
$$= (-3B_{123} + \frac{1}{2}(B_{144}B_{324} + B_{244}B_{314} - B_{344}B_{214}))$$
$$+\frac{\alpha_s}{8\pi^2}\int d\vec r_0 \left(A_{34}\frac{\vec r_{34}^{\,2}}{\vec r_{03}^{\,2}\vec r_{04}^{\,2}}\ln\left(\frac{\vec r_{34}^{\,2}a}{\vec r_{03}^{\,2}\vec r_{04}^{\,2}}\right) + A_{13}\frac{\vec r_{13}^{\,2}}{\vec r_{03}^{\,2}\vec r_{01}^{\,2}}\ln\left(\frac{\vec r_{13}^{\,2}a}{\vec r_{03}^{\,2}\vec r_{01}^{\,2}}\right) + A_{23}\frac{\vec r_{23}^{\,2}}{\vec r_{03}^{\,2}\vec r_{02}^{\,2}}\ln\left(\frac{\vec r_{23}^{\,2}a}{\vec r_{03}^{\,2}\vec r_{02}^{\,2}}\right)\right.$$
$$\left.+A_{14}\frac{\vec r_{14}^{\,2}}{\vec r_{01}^{\,2}\vec r_{04}^{\,2}}\ln\left(\frac{\vec r_{14}^{\,2}a}{\vec r_{01}^{\,2}\vec r_{04}^{\,2}}\right) + A_{24}\frac{\vec r_{24}^{\,2}}{\vec r_{02}^{\,2}\vec z_{04}^{\,2}}\ln\left(\frac{\vec r_{24}^{\,2}a}{\vec r_{02}^{\,2}\vec z_{04}^{\,2}}\right) + A_{12}\frac{\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}\ln\left(\frac{\vec r_{12}^{\,2}a}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}\right)\right), \tag{4.5}$$

where the functions A are calculated in appendix B (B.4–B.8) according to model (4.1). Therefore the evolution equation for B_{123}^{conf} turns into

$$\frac{\partial B_{123}^{conf}}{\partial\eta} = \frac{\alpha_s 3}{4\pi^2}\int d\vec r_4 \left[\frac{\vec r_{12}^{\,2}}{\vec r_{41}^{\,2}\vec r_{42}^{\,2}}(-B_{123} + \frac{1}{6}(B_{144}B_{324} + B_{244}B_{314} - B_{344}B_{214}))^{conf}\right.$$

$$+ (1\leftrightarrow 3) + (2\leftrightarrow 3)\Big] - \frac{\alpha_s^2}{8\pi^4}\int d\vec r_0 d\vec r_4\, \mathbf{G}_{finite} - \frac{\alpha_s^2}{8\pi^3}\int d\vec r_0\, \mathbf{G}'$$

$$-\frac{\alpha_s}{4\pi^2}\frac{\alpha_s}{8\pi^2}\int d\vec r_4 d\vec r_0 \left[\frac{\vec r_{12}^{\,2}}{\vec r_{41}^{\,2}\vec r_{42}^{\,2}}\left(A_{34}\frac{\vec r_{34}^{\,2}}{\vec r_{03}^{\,2}\vec r_{04}^{\,2}}\ln\left(\frac{\vec r_{34}^{\,2}a}{\vec r_{03}^{\,2}\vec r_{04}^{\,2}}\right) + A_{13}\frac{\vec r_{13}^{\,2}}{\vec z_{03}^{\,2}\vec r_{01}^{\,2}}\ln\left(\frac{\vec r_{13}^{\,2}a}{\vec z_{03}^{\,2}\vec r_{01}^{\,2}}\right)\right.\right.$$

$$+A_{23}\frac{\vec r_{23}^{\,2}}{\vec r_{03}^{\,2}\vec r_{02}^{\,2}}\ln\left(\frac{\vec r_{23}^{\,2}a}{\vec r_{03}^{\,2}\vec r_{02}^{\,2}}\right) + A_{14}\frac{\vec r_{14}^{\,2}}{\vec r_{01}^{\,2}\vec r_{04}^{\,2}}\ln\left(\frac{\vec r_{14}^{\,2}a}{\vec r_{01}^{\,2}\vec r_{04}^{\,2}}\right) + A_{24}\frac{\vec r_{24}^{\,2}}{\vec r_{02}^{\,2}\vec z_{04}^{\,2}}\ln\left(\frac{\vec r_{24}^{\,2}a}{\vec r_{02}^{\,2}\vec z_{04}^{\,2}}\right)$$

$$\left.\left.+A_{12}\frac{\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}\ln\left(\frac{\vec r_{12}^{\,2}a}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}\right)\right) + (1\leftrightarrow 3) + (2\leftrightarrow 3)\right]$$

$$+\frac{\alpha_s}{8\pi^2}\frac{\alpha_s}{4\pi^2}\int d\vec r_4 d\vec r_0 \left[\frac{\vec r_{12}^{\,2}}{\vec r_{41}^{\,2}\vec z_{42}^{\,2}}\ln\left(\frac{a\vec r_{12}^{\,2}}{\vec r_{41}^{\,2}\vec r_{42}^{\,2}}\right)\right.$$

$$\times\left(A_{34}\frac{\vec r_{34}^{\,2}}{\vec r_{03}^{\,2}\vec z_{04}^{\,2}} + A_{13}\frac{\vec r_{13}^{\,2}}{\vec r_{03}^{\,2}\vec z_{01}^{\,2}} + A_{23}\frac{\vec r_{23}^{\,2}}{\vec r_{03}^{\,2}\vec r_{02}^{\,2}} + A_{14}\frac{\vec r_{14}^{\,2}}{\vec r_{01}^{\,2}\vec r_{04}^{\,2}} + A_{24}\frac{\vec r_{24}^{\,2}}{\vec r_{02}^{\,2}\vec r_{04}^{\,2}} + A_{12}\frac{\vec r_{12}^{\,2}}{\vec z_{01}^{\,2}\vec r_{02}^{\,2}}\right)$$

$$\left. + (1\leftrightarrow 3) + (2\leftrightarrow 3)\right]. \tag{4.6}$$

After simplification one has

$$\frac{\partial B_{123}^{conf}}{\partial\eta} = \frac{\alpha_s 3}{4\pi^2}\int d\vec r_4 \left[\frac{\vec r_{12}^{\,2}}{\vec r_{41}^{\,2}\vec r_{42}^{\,2}}(-B_{123} + \frac{1}{6}(B_{144}B_{324} + B_{244}B_{314} - B_{344}B_{214}))^{conf}\right.$$

$$+ (1\leftrightarrow 3) + (2\leftrightarrow 3)\Big] - \frac{\alpha_s^2}{8\pi^4}\int d\vec r_0 d\vec r_4\, \mathbf{G}_{finite} - \frac{\alpha_s^2}{8\pi^3}\int d\vec r_0\, \mathbf{G}'$$

$$-\frac{\alpha_s^2}{32\pi^4}\int d\vec r_4 d\vec r_0 \left[\frac{\vec r_{12}^{\,2}}{\vec r_{41}^{\,2}\vec r_{42}^{\,2}}\left(A_{34}\frac{\vec r_{34}^{\,2}}{\vec r_{03}^{\,2}\vec r_{04}^{\,2}}\ln\left(\frac{\vec r_{34}^{\,2}\vec r_{41}^{\,2}\vec r_{42}^{\,2}}{\vec r_{03}^{\,2}\vec r_{04}^{\,2}\vec r_{12}^{\,2}}\right) + A_{13}\frac{\vec r_{13}^{\,2}}{\vec r_{03}^{\,2}\vec r_{01}^{\,2}}\ln\left(\frac{\vec r_{13}^{\,2}\vec r_{41}^{\,2}\vec r_{42}^{\,2}}{\vec r_{03}^{\,2}\vec r_{01}^{\,2}\vec r_{12}^{\,2}}\right)\right.\right.$$

$$+A_{23}\frac{\vec r_{23}^{\,2}}{\vec r_{03}^{\,2}\vec r_{02}^{\,2}}\ln\left(\frac{\vec r_{23}^{\,2}\vec r_{41}^{\,2}\vec r_{42}^{\,2}}{\vec r_{03}^{\,2}\vec r_{02}^{\,2}\vec r_{12}^{\,2}}\right) + A_{14}\frac{\vec r_{14}^{\,2}}{\vec r_{01}^{\,2}\vec r_{04}^{\,2}}\ln\left(\frac{\vec r_{14}^{\,4}\vec r_{42}^{\,2}}{\vec r_{01}^{\,2}\vec r_{04}^{\,2}\vec r_{12}^{\,2}}\right) + A_{24}\frac{\vec r_{24}^{\,2}}{\vec r_{02}^{\,2}\vec r_{04}^{\,2}}\ln\left(\frac{\vec r_{24}^{\,4}\vec r_{41}^{\,2}}{\vec r_{02}^{\,2}\vec r_{04}^{\,2}\vec r_{12}^{\,2}}\right)$$

$$\left.\left.+A_{12}\frac{\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec z_{02}^{\,2}}\ln\left(\frac{\vec r_{41}^{\,2}\vec r_{42}^{\,2}}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}\right)\right) + (1\leftrightarrow 3) + (2\leftrightarrow 3)\right]. \tag{4.7}$$

Now we can symmetrize the last 3 lines of this expression w.r.t. $0\leftrightarrow 4$ transformation. After that one can use (A.9) to show that all the nonconformal terms have the $SU(3)$ coefficients independent either of $\vec r_4$ or of $\vec r_0$.

So first we add the symmetrized last 3 lines of the previous expression to the nonconformal part of $\mathbf{G}_{finite}$ (3.20). Taking into account (A.3), (A.9), and (A.13), we have

$$-\frac{\alpha_s^2}{8\pi^4}\int d\vec r_0 d\vec r_4 \tilde{\mathbf{G}} = -\frac{\alpha_s^2}{8\pi^4}\int d\vec r_0 d\vec r_4 \left[\Big\{(M_{13} - M_{12} - M_{23} + M_2)[(U_0 U_4{}^\dagger U_3)\cdot(U_2 U_0{}^\dagger U_1)\cdot U_4\right.$$

$$\left.+(U_1 U_0{}^\dagger U_2)\cdot(U_3 U_4{}^\dagger U_0)\cdot U_4] + (\text{all 5 permutations } 1\leftrightarrow 2\leftrightarrow 3)\Big\} + (0\leftrightarrow 4)\right]$$

$$-\frac{\alpha_s^2}{32\pi^4}\int d\vec r_4 d\vec r_0 \left[\frac{\vec r_{12}^{\,2}}{\vec r_{41}^{\,2}\vec r_{42}^{\,2}}\left(A_{34}\frac{\vec r_{34}^{\,2}}{\vec r_{03}^{\,2}\vec r_{04}^{\,2}}\ln\left(\frac{\vec r_{34}^{\,2}\vec r_{41}^{\,2}\vec r_{42}^{\,2}}{\vec r_{03}^{\,2}\vec r_{04}^{\,2}\vec r_{12}^{\,2}}\right)\right.\right.$$

$$+A_{23}\frac{\vec r_{23}^{\,2}}{\vec r_{03}^{\,2}\vec r_{02}^{\,2}}\ln\left(\frac{\vec r_{23}^{\,2}\vec r_{41}^{\,2}\vec r_{42}^{\,2}}{\vec r_{03}^{\,2}\vec r_{02}^{\,2}\vec r_{12}^{\,2}}\right) + A_{14}\frac{\vec r_{14}^{\,2}}{\vec r_{01}^{\,2}\vec r_{04}^{\,2}}\ln\left(\frac{\vec r_{14}^{\,4}\vec r_{42}^{\,2}}{\vec r_{01}^{\,2}\vec r_{04}^{\,2}\vec r_{12}^{\,2}}\right) + A_{24}\frac{\vec r_{24}^{\,2}}{\vec r_{02}^{\,2}\vec r_{04}^{\,2}}\ln\left(\frac{\vec r_{24}^{\,4}\vec r_{41}^{\,2}}{\vec r_{02}^{\,2}\vec r_{04}^{\,2}\vec r_{12}^{\,2}}\right)$$

$$+A_{13}\frac{\vec r_{13}^{\,2}}{\vec r_{03}^{\,2}\vec r_{01}^{\,2}}\ln\left(\frac{\vec r_{13}^{\,2}\vec r_{41}^{\,2}\vec r_{42}^{\,2}}{\vec r_{03}^{\,2}\vec r_{01}^{\,2}\vec r_{12}^{\,2}}\right)+A_{12}\frac{\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec z_{02}^{\,2}}\ln\left(\frac{\vec r_{41}^{\,2}\vec r_{42}^{\,2}}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}\right)\Bigg)+(1\leftrightarrow 3)+(2\leftrightarrow 3)\Bigg]$$

$$=-\frac{\alpha_s^2}{8\pi^4}\int d\vec r_0 d\vec r_4\left(\frac{\vec r_{12}{}^4}{8\vec r_{01}{}^2\vec r_{02}{}^2\vec r_{14}{}^2\vec r_{24}{}^2}\ln\left(\frac{\vec r_{14}{}^2\vec r_{24}{}^2}{\vec r_{01}{}^2\vec r_{02}{}^2}\right)\left(\frac{1}{2}B_{003}B_{012}-2B_{001}B_{023}\right)\right.$$

$$+\frac{\vec r_{12}{}^2}{8\vec r_{01}{}^2\vec r_{02}{}^2}\left(\frac{\vec r_{13}{}^2}{\vec r_{14}{}^2\vec r_{34}{}^2}\ln\left(\frac{\vec r_{12}{}^2\vec r_{14}{}^2\vec r_{34}{}^2}{\vec r_{01}{}^2\vec r_{02}{}^2\vec r_{13}{}^2}\right)+\frac{\vec r_{03}{}^2}{\vec r_{04}{}^2\vec r_{34}{}^2}\ln\left(\frac{\vec r_{01}{}^2\vec r_{02}{}^2\vec r_{03}{}^2}{\vec r_{04}{}^2\vec r_{12}{}^2\vec r_{34}{}^2}\right)\right)$$

$$\times\left(B_{003}B_{012}-B_{001}B_{023}\right)$$

$$+\frac{\vec r_{12}{}^2}{8\vec r_{02}{}^2\vec r_{04}{}^2\vec r_{14}{}^2}\left[\ln\left(\frac{\vec r_{01}{}^4\vec r_{02}{}^2}{\vec r_{04}{}^2\vec r_{12}{}^2\vec r_{14}{}^2}\right)\left(2B_{003}B_{012}-2B_{002}B_{013}-3B_{001}B_{023}+4B_{123}\right)\right.$$

$$+\ln\left(\frac{\vec r_{04}{}^2\vec r_{12}{}^2}{\vec r_{01}{}^2\vec r_{24}{}^2}\right)\left(-tr\left(U_0{}^\dagger U_4\right)\left(U_1U_4{}^\dagger U_2+U_2U_4{}^\dagger U_1\right)\cdot U_0\cdot U_3\right.$$

$$\left.+2\left(U_1\cdot U_2\cdot U_3-\left(U_0U_4{}^\dagger U_1\right)\cdot\left(U_2U_0{}^\dagger U_4\right)\cdot U_3\right)\right)$$

$$+\ln\left(\frac{\vec r_{01}{}^4\vec r_{02}{}^2\vec r_{34}{}^4}{\vec r_{03}{}^4\vec r_{04}{}^2\vec r_{12}{}^2\vec r_{14}{}^2}\right)\left(tr\left(U_4{}^\dagger U_1\right)\left(\left(U_2U_0{}^\dagger U_4\right)\cdot U_0\cdot U_3+\left(U_4U_0{}^\dagger U_2\right)\cdot U_0\cdot U_3\right)\right.$$

$$\left.\left.+\left(U_0U_4{}^\dagger U_1U_0{}^\dagger U_2\right)\cdot U_3\cdot U_4+\left(U_2U_0{}^\dagger U_1U_4{}^\dagger U_0\right)\cdot U_3\cdot U_4\right)\right]$$

$$+\left(\frac{\vec r_{12}{}^2\vec r_{13}{}^2}{16\vec r_{01}{}^2\vec r_{02}{}^2\vec r_{14}{}^2\vec r_{34}{}^2}\ln\left(\frac{\vec r_{01}{}^2\vec r_{12}{}^2\vec r_{34}{}^2}{\vec r_{02}{}^2\vec r_{13}{}^2\vec r_{14}{}^2}\right)+\frac{\vec r_{03}{}^2\vec r_{12}{}^2}{8\vec r_{01}{}^2\vec r_{02}{}^2\vec r_{04}{}^2\vec r_{34}{}^2}\ln\left(\frac{\vec r_{02}{}^2\vec r_{03}{}^2\vec r_{14}{}^4}{\vec r_{01}{}^2\vec r_{04}{}^2\vec r_{12}{}^2\vec r_{34}{}^2}\right)\right)$$

$$\times\left(\left(U_0U_4{}^\dagger U_3\right)\cdot\left(U_1U_0{}^\dagger U_2\right)\cdot U_4+\left(U_2U_0{}^\dagger U_1\right)\cdot\left(U_3U_4{}^\dagger U_0\right)\cdot U_4\right.$$

$$\left.-\left(U_2U_0{}^\dagger U_4\right)\cdot\left(U_3U_4{}^\dagger U_1\right)\cdot U_0-\left(U_1U_4{}^\dagger U_3\right)\cdot\left(U_4U_0{}^\dagger U_2\right)\cdot U_0\right)$$

$$\left.+\left(\text{all 5 permutations}\,1\leftrightarrow 2\leftrightarrow 3\right)\right)+(0\leftrightarrow 4). \tag{4.8}$$

Indeed, in this expression all the nonconformal terms have the $SU(3)$ coefficients independent either of $\vec r_4$ or of $\vec r_0$. Therefore one can integrate them w.r.t. $\vec r_4$ or $\vec r_0$. However, it is easier to transform (3.30) using integral (116) from [6]. We use it in the symmetric form

$$\frac{\vec r_{12}^{\,2}}{\vec r_{10}^{\,2}\vec r_{20}^{\,2}}\ln\frac{\vec r_{10}^{\,2}}{\vec r_{12}^{\,2}}\ln\frac{\vec r_{20}^{\,2}}{\vec r_{12}^{\,2}}=2\pi\zeta\left(3\right)\left(\delta\left(\vec r_{10}\right)+\delta\left(\vec r_{20}\right)\right)$$

$$+\frac{\vec r_{12}^{\,2}}{\vec r_{10}^{\,2}\vec r_{20}^{\,2}}\int\frac{d\vec r_4}{2\pi}\left(\frac{\vec r_{20}^{\,2}}{\vec r_{04}^{\,2}\vec r_{24}^{\,2}}+\frac{\vec r_{10}^{\,2}}{\vec r_{04}^{\,2}\vec r_{14}^{\,2}}-\frac{\vec r_{12}^{\,2}}{\vec r_{24}^{\,2}\vec r_{14}^{\,2}}\right)\ln\left(\frac{\vec r_{10}^{\,2}\vec r_{20}^{\,2}}{\vec r_{14}^{\,2}\vec r_{24}^{\,2}}\right). \tag{4.9}$$

Then,

$$\mathbf{G}'=\frac{1}{2}\left[\frac{\vec r_{13}^{\,2}\vec r_{20}^{\,2}}{\vec r_{30}^{\,2}\vec r_{12}^{\,2}}-\frac{\vec r_{32}^{\,2}\vec r_{10}^{\,2}}{\vec r_{30}^{\,2}\vec r_{12}^{\,2}}\right]\frac{\vec r_{12}^{\,2}}{\vec r_{10}^{\,2}\vec r_{20}^{\,2}}\int\frac{d\vec r_4}{2\pi}\left(\frac{\vec r_{20}^{\,2}}{\vec r_{04}^{\,2}\vec r_{24}^{\,2}}+\frac{\vec r_{10}^{\,2}}{\vec r_{04}^{\,2}\vec r_{14}^{\,2}}-\frac{\vec r_{12}^{\,2}}{\vec r_{24}^{\,2}\vec r_{14}^{\,2}}\right)\ln\left(\frac{\vec r_{10}^{\,2}\vec r_{20}^{\,2}}{\vec r_{14}^{\,2}\vec r_{24}^{\,2}}\right)$$

$$\times\left(B_{100}B_{320}-B_{200}B_{310}\right)$$

$$+\left[\frac{\vec r_{13}^{\,2}\vec r_{20}^{\,2}}{\vec r_{30}^{\,2}\vec r_{12}^{\,2}}-\frac{\vec r_{32}^{\,2}\vec r_{10}^{\,2}}{\vec r_{30}^{\,2}\vec r_{12}^{\,2}}\right]\zeta\left(3\right)\pi\left(\delta\left(\vec r_{10}\right)+\delta\left(\vec r_{20}\right)\right)\left(B_{100}B_{320}-B_{200}B_{310}\right)$$

$$-\frac{\vec{r}_{12}^{2}}{\vec{r}_{10}^{2}\vec{r}_{20}^{\,2}}\int\frac{d\vec{r}_4}{2\pi}\left(\frac{\vec{r}_{20}^{2}}{\vec{r}_{04}^{2}\vec{r}_{24}^{\,2}}+\frac{\vec{r}_{10}^{2}}{\vec{r}_{04}^{2}\vec{r}_{14}^{\,2}}-\frac{\vec{r}_{12}^{2}}{\vec{r}_{24}^{2}\vec{r}_{14}^{\,2}}\right)\ln\left(\frac{\vec{r}_{10}^{2}\vec{r}_{20}^{2}}{\vec{r}_{14}^{2}\vec{r}_{24}^{2}}\right)$$
$$\times\left(9B_{123}-\frac{1}{2}\left[2\left(B_{100}B_{320}+B_{200}B_{130}\right)-B_{300}B_{120}\right]\right)$$
$$-2\pi\zeta\left(3\right)\left(\delta\left(\vec{r}_{10}\right)+\delta\left(\vec{r}_{20}\right)\right)\left(9B_{123}-\frac{1}{2}\left[2\left(B_{100}B_{320}+B_{200}B_{130}\right)-B_{300}B_{120}\right]\right)$$
$$+\frac{11}{6}\left[\ln\left(\frac{\vec{r}_{01}^{\,2}}{\vec{r}_{02}^{\,2}}\right)\left(\frac{1}{\vec{r}_{02}^{\,2}}-\frac{1}{\vec{r}_{01}^{\,2}}\right)-\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}}\ln\left(\frac{\vec{r}_{12}^{\,2}}{\tilde{\mu}^{2}}\right)\right]$$
$$\times\left(\frac{3}{2}(B_{100}B_{230}+B_{200}B_{130}-B_{300}B_{210})-9B_{123}\right)+(1\leftrightarrow 3)+(2\leftrightarrow 3). \tag{4.10}$$

One can write it as

$$\mathbf{G}'=\frac{1}{2}\left[\frac{\vec{r}_{13}^{\,2}\vec{r}_{20}^{\,2}}{\vec{r}_{30}^{\,2}\vec{r}_{12}^{2}}-\frac{\vec{r}_{32}^{2}\vec{r}_{10}^{2}}{\vec{r}_{30}^{2}\vec{r}_{12}^{2}}\right]\frac{\vec{r}_{12}^{2}}{\vec{r}_{10}^{2}\vec{r}_{20}^{\,2}}\int\frac{d\vec{r}_4}{2\pi}\left(\frac{\vec{r}_{20}^{2}}{\vec{r}_{04}^{2}\vec{r}_{24}^{\,2}}+\frac{\vec{r}_{10}^{2}}{\vec{r}_{04}^{2}\vec{r}_{14}^{\,2}}-\frac{\vec{r}_{12}^{2}}{\vec{r}_{24}^{2}\vec{r}_{14}^{\,2}}\right)\ln\left(\frac{\vec{r}_{10}^{2}\vec{r}_{20}^{2}}{\vec{r}_{14}^{2}\vec{r}_{24}^{2}}\right)$$
$$\times\left(B_{100}B_{320}-B_{200}B_{310}\right)-\frac{\vec{r}_{12}^{2}}{\vec{r}_{10}^{2}\vec{r}_{20}^{\,2}}\int\frac{d\vec{r}_4}{2\pi}\left(\frac{\vec{r}_{20}^{2}}{\vec{r}_{04}^{2}\vec{r}_{24}^{\,2}}+\frac{\vec{r}_{10}^{2}}{\vec{r}_{04}^{2}\vec{r}_{14}^{\,2}}-\frac{\vec{r}_{12}^{2}}{\vec{r}_{24}^{2}\vec{r}_{14}^{\,2}}\right)\ln\left(\frac{\vec{r}_{10}^{2}\vec{r}_{20}^{2}}{\vec{r}_{14}^{2}\vec{r}_{24}^{2}}\right)$$
$$\times\left(9B_{123}-\frac{1}{2}\left[2\left(B_{100}B_{320}+B_{200}B_{130}\right)-B_{300}B_{120}\right]\right)$$
$$+\frac{11}{6}\left[\ln\left(\frac{\vec{r}_{01}^{\,2}}{\vec{r}_{02}^{\,2}}\right)\left(\frac{1}{\vec{r}_{02}^{\,2}}-\frac{1}{\vec{r}_{01}^{\,2}}\right)-\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}}\ln\left(\frac{\vec{r}_{12}^{\,2}}{\tilde{\mu}^{2}}\right)\right]$$
$$\times\left(\frac{3}{2}(B_{100}B_{230}+B_{200}B_{130}-B_{300}B_{210})-9B_{123}\right)+(1\leftrightarrow 3)+(2\leftrightarrow 3). \tag{4.11}$$

Next, we symmetrize the previous expression w.r.t. $0\leftrightarrow 4$ exchange and combine it with (3.20), (4.7), and (4.8) to obtain the NLO kernel for the composite 3QWL operator B_{123}^{conf}

$$\langle K_{NLO}\otimes B_{123}^{conf}\rangle=-\frac{\alpha_s^2}{8\pi^3}\int d\vec{r}_0\mathbf{G}'-\frac{\alpha_s^2}{8\pi^4}\int d\vec{r}_0 d\vec{r}_4\tilde{\mathbf{G}}$$
$$-\frac{\alpha_s^2}{8\pi^4}\int d\vec{r}_0 d\vec{r}_4\ \left(\left\{\tilde{L}_{12}\left(U_0U_4{}^{\dagger}U_2\right)\cdot\left(U_1U_0{}^{\dagger}U_4\right)\cdot U_3\right.\right.$$
$$+L_{12}\left[\left(U_0U_4{}^{\dagger}U_2\right)\cdot\left(U_1U_0{}^{\dagger}U_4\right)\cdot U_3+tr\left(U_0U_4{}^{\dagger}\right)\left(U_1U_0{}^{\dagger}U_2\right)\cdot U_3\cdot U_4\right.$$
$$\left.-\frac{3}{4}[B_{144}B_{234}+B_{244}B_{134}-B_{344}B_{124}]+\frac{1}{2}B_{123}\right]$$
$$\left.\left.+\,(\text{all 5 permutations}\,1\leftrightarrow 2\leftrightarrow 3)\right\}+(0\leftrightarrow 4)\right). \tag{4.12}$$

Using (A.9) to get rid of the terms like

$$\left(U_0U_4{}^{\dagger}U_1U_0{}^{\dagger}U_2\right)\cdot U_3\cdot U_4+\left(U_2U_0{}^{\dagger}U_1U_4{}^{\dagger}U_0\right)\cdot U_3\cdot U_4 \tag{4.13}$$

it can be transformed to

$$\langle K_{NLO}\otimes B_{123}^{conf}\rangle=-\frac{\alpha_s^2}{8\pi^4}\int d\vec{r}_0 d\vec{r}_4\ \left(\left\{\tilde{L}_{12}^{C}\left(U_0U_4{}^{\dagger}U_2\right)\cdot\left(U_1U_0{}^{\dagger}U_4\right)\cdot U_3\right.\right.$$

$$
\begin{aligned}
&+L^C_{12}\left[\left(U_0U_4{}^\dagger U_2\right)\cdot\left(U_1U_0{}^\dagger U_4\right)\cdot U_3+tr\left(U_0U_4{}^\dagger\right)\left(U_1U_0{}^\dagger U_2\right)\cdot U_3\cdot U_4\right.\\
&\qquad\left.-\frac{3}{4}[B_{144}B_{234}+B_{244}B_{134}-B_{344}B_{124}]+\frac{1}{2}B_{123}\right]\\
&+M^C_{12}\left[\left(U_0U_4{}^\dagger U_3\right)\cdot\left(U_2U_0{}^\dagger U_1\right)\cdot U_4+\left(U_1U_0{}^\dagger U_2\right)\cdot\left(U_3U_4{}^\dagger U_0\right)\cdot U_4\right]\\
&+Z_{12}B_{003}B_{012}+\ \text{(all 5 permutations}\,1\leftrightarrow 2\leftrightarrow 3)\Big\}+(0\leftrightarrow 4)\Big)\\
&-\frac{\alpha_s^2}{8\pi^3}\int d\vec{r}_0\left(\frac{11}{6}\left[\ln\left(\frac{\vec{r}_{01}^{\,2}}{\vec{r}_{02}^{\,2}}\right)\left(\frac{1}{\vec{r}_{02}^{\,2}}-\frac{1}{\vec{r}_{01}^{\,2}}\right)-\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}}\ln\left(\frac{\vec{r}_{12}^{\,2}}{\tilde{\mu}^2}\right)\right]\right.\\
&\times\left(\frac{3}{2}(B_{100}B_{230}+B_{200}B_{130}-B_{300}B_{210})-9B_{123}\right)+(1\leftrightarrow 3)+(2\leftrightarrow 3)\Bigg). \qquad (4.14)
\end{aligned}
$$

Here

$$
L^C_{12}=L_{12}+\frac{\vec{r}_{12}{}^2}{4\vec{r}_{01}{}^2\vec{r}_{04}{}^2\vec{r}_{24}{}^2}\ln\left(\frac{\vec{r}_{02}{}^2\vec{r}_{14}{}^2}{\vec{r}_{04}{}^2\vec{r}_{12}{}^2}\right)+\frac{\vec{r}_{12}{}^2}{4\vec{r}_{02}{}^2\vec{r}_{04}{}^2\vec{r}_{14}{}^2}\ln\left(\frac{\vec{r}_{01}{}^2\vec{r}_{24}{}^2}{\vec{r}_{04}{}^2\vec{r}_{12}{}^2}\right), \qquad (4.15)
$$

$$
\tilde{L}^C_{12}=\tilde{L}_{12}+\frac{\vec{r}_{12}{}^2}{4\vec{r}_{01}{}^2\vec{r}_{04}{}^2\vec{r}_{24}{}^2}\ln\left(\frac{\vec{r}_{02}{}^2\vec{r}_{14}{}^2}{\vec{r}_{04}{}^2\vec{r}_{12}{}^2}\right)-\frac{\vec{r}_{12}{}^2}{4\vec{r}_{02}{}^2\vec{r}_{04}{}^2\vec{r}_{14}{}^2}\ln\left(\frac{\vec{r}_{01}{}^2\vec{r}_{24}{}^2}{\vec{r}_{04}{}^2\vec{r}_{12}{}^2}\right), \qquad (4.16)
$$

$$
\begin{aligned}
M^C_{12}=&\frac{\vec{r}_{12}{}^2}{16\vec{r}_{02}{}^2\vec{r}_{04}{}^2\vec{r}_{14}{}^2}\ln\left(\frac{\vec{r}_{01}{}^2\vec{r}_{02}{}^2\vec{r}_{34}{}^4}{\vec{r}_{03}{}^4\vec{r}_{14}{}^2\vec{r}_{24}{}^2}\right)+\frac{\vec{r}_{12}{}^2}{16\vec{r}_{01}{}^2\vec{r}_{04}{}^2\vec{r}_{24}{}^2}\ln\left(\frac{\vec{r}_{03}{}^4\vec{r}_{04}{}^4\vec{r}_{12}{}^4\vec{r}_{24}{}^2}{\vec{r}_{01}{}^2\vec{r}_{02}{}^6\vec{r}_{14}{}^2\vec{r}_{34}{}^4}\right)\\
&+\frac{\vec{r}_{23}{}^2}{16\vec{r}_{02}{}^2\vec{r}_{04}{}^2\vec{r}_{34}{}^2}\ln\left(\frac{\vec{r}_{01}{}^4\vec{r}_{03}{}^2\vec{r}_{24}{}^6\vec{r}_{34}{}^2}{\vec{r}_{02}{}^2\vec{r}_{04}{}^4\vec{r}_{14}{}^4\vec{r}_{23}{}^4}\right)+\frac{\vec{r}_{23}{}^2}{16\vec{r}_{03}{}^2\vec{r}_{04}{}^2\vec{r}_{24}{}^2}\ln\left(\frac{\vec{r}_{02}{}^2\vec{r}_{03}{}^2\vec{r}_{14}{}^4}{\vec{r}_{01}{}^4\vec{r}_{24}{}^2\vec{r}_{34}{}^2}\right)\\
&+\frac{\vec{r}_{13}{}^2}{16\vec{r}_{03}{}^2\vec{r}_{04}{}^2\vec{r}_{14}{}^2}\ln\left(\frac{\vec{r}_{02}{}^4\vec{r}_{14}{}^2\vec{r}_{34}{}^2}{\vec{r}_{01}{}^2\vec{r}_{03}{}^2\vec{r}_{24}{}^4}\right)+\frac{\vec{r}_{13}{}^2}{16\vec{r}_{01}{}^2\vec{r}_{04}{}^2\vec{r}_{34}{}^2}\ln\left(\frac{\vec{r}_{02}{}^4\vec{r}_{14}{}^2\vec{r}_{34}{}^2}{\vec{r}_{01}{}^2\vec{r}_{03}{}^2\vec{r}_{24}{}^4}\right)\\
&+\frac{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}{8\vec{r}_{01}{}^2\vec{r}_{02}{}^2\vec{r}_{04}{}^2\vec{r}_{34}{}^2}\ln\left(\frac{\vec{r}_{01}{}^2\vec{r}_{03}{}^2\vec{r}_{24}{}^4}{\vec{r}_{02}{}^2\vec{r}_{04}{}^2\vec{r}_{12}{}^2\vec{r}_{34}{}^2}\right)+\frac{\vec{r}_{23}{}^2\vec{r}_{12}{}^2}{8\vec{r}_{01}{}^2\vec{r}_{02}{}^2\vec{r}_{24}{}^2\vec{r}_{34}{}^2}\ln\left(\frac{\vec{r}_{02}{}^2\vec{r}_{12}{}^2\vec{r}_{34}{}^2}{\vec{r}_{01}{}^2\vec{r}_{23}{}^2\vec{r}_{24}{}^2}\right)\\
&+\frac{\vec{r}_{14}{}^2\vec{r}_{23}{}^2}{8\vec{r}_{01}{}^2\vec{r}_{04}{}^2\vec{r}_{24}{}^2\vec{r}_{34}{}^2}\ln\left(\frac{\vec{r}_{01}{}^2\vec{r}_{04}{}^2\vec{r}_{23}{}^2\vec{r}_{24}{}^2}{\vec{r}_{02}{}^4\vec{r}_{14}{}^2\vec{r}_{34}{}^2}\right), \qquad (4.17)
\end{aligned}
$$

$$
\begin{aligned}
Z_{12}=&\frac{\vec{r}_{12}{}^2}{8\vec{r}_{01}{}^2\vec{r}_{02}{}^2}\left[\left(\frac{\vec{r}_{03}{}^2}{\vec{r}_{04}{}^2\vec{r}_{34}{}^2}-\frac{\vec{r}_{02}{}^2}{\vec{r}_{04}{}^2\vec{r}_{24}{}^2}\right)\ln\left(\frac{\vec{r}_{02}{}^2\vec{r}_{14}{}^2}{\vec{r}_{04}{}^2\vec{r}_{12}{}^2}\right)\right.\\
&\left.+\frac{\vec{r}_{01}{}^2}{\vec{r}_{04}{}^2\vec{r}_{14}{}^2}\ln\left(\frac{\vec{r}_{02}{}^2\vec{r}_{34}{}^2}{\vec{r}_{03}{}^2\vec{r}_{24}{}^2}\right)+\frac{\vec{r}_{13}{}^2}{\vec{r}_{14}{}^2\vec{r}_{34}{}^2}\ln\left(\frac{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}\right)\right]-(1\leftrightarrow 3), \qquad (4.18)
\end{aligned}
$$

and L_{12} and $\tilde{L}_{12}$ are the elements of the nonconformal kernel defined in (3.9) and (3.10). Checking that L^C_{12}, $\tilde{L}^C_{12}$, M^C_{12}, and Z_{12} have integrable singularities at $\vec{r}_4=\vec{r}_0$ and that L^C_{12}, $\tilde{L}^C_{12}$, and Z_{12} have integrable singularities at $\vec{r}_4=\vec{r}_{1,2,3}$ is straightforward. To prove that all the terms with M^C have safe behavior at $\vec{r}_4=\vec{r}_{1,2,3}$ one has to use $SU(3)$ identity (A.14).

Now one can see that the NLO kernel for the evolution equation for the composite 3QWL operator B^{conf}_{123} (4.2) is quasi-conformal if one expresses the LO kernel in terms of composite operator (4.5).

The term with Z can be integrated w.r.t. $\vec{r}_4$. The integral is calculated in the appendix C

$$\int \frac{d\vec{r}_4}{\pi} Z_{12} = \frac{\vec{r}_{32}{}^2}{8\vec{r}_{03}{}^2\vec{r}_{02}{}^2}\ln^2\left(\frac{\vec{r}_{32}{}^2\vec{r}_{10}{}^2}{\vec{r}_{13}{}^2\vec{r}_{20}{}^2}\right) - \frac{\vec{r}_{12}{}^2}{8\vec{r}_{01}{}^2\vec{r}_{02}{}^2}\ln^2\left(\frac{\vec{r}_{12}{}^2\vec{r}_{30}{}^2}{\vec{r}_{13}{}^2\vec{r}_{20}{}^2}\right). \tag{4.19}$$

Finally, the kernel reads

$$\begin{aligned}
\langle K_{NLO}\otimes B_{123}^{conf}\rangle = -\frac{\alpha_s^2}{8\pi^4}\int d\vec{r}_0 d\vec{r}_4 \;\; &\left(\left\{\tilde{L}_{12}^C\left(U_0U_4{}^\dagger U_2\right)\cdot\left(U_1U_0{}^\dagger U_4\right)\cdot U_3\right.\right.\\
&+L_{12}^C\left[\left(U_0U_4{}^\dagger U_2\right)\cdot\left(U_1U_0{}^\dagger U_4\right)\cdot U_3 + tr\left(U_0U_4{}^\dagger\right)\left(U_1U_0{}^\dagger U_2\right)\cdot U_3\cdot U_4\right.\\
&\left.-\frac{3}{4}[B_{144}B_{234}+B_{244}B_{134}-B_{344}B_{124}]+\frac{1}{2}B_{123}\right]\\
&+M_{12}^C\left[\left(U_0U_4{}^\dagger U_3\right)\cdot\left(U_2U_0{}^\dagger U_1\right)\cdot U_4+\left(U_1U_0{}^\dagger U_2\right)\cdot\left(U_3U_4{}^\dagger U_0\right)\cdot U_4\right]\\
&\left.\left.+\,(\text{all 5 permutations}\,1\leftrightarrow 2\leftrightarrow 3)\right\}+(0\leftrightarrow 4)\right)\\
-\frac{\alpha_s^2}{8\pi^3}\int d\vec{r}_0 &\left(\frac{11}{6}\left[\ln\left(\frac{\vec{r}_{01}^{\,2}}{\vec{r}_{02}^{\,2}}\right)\left(\frac{1}{\vec{r}_{02}^{\,2}}-\frac{1}{\vec{r}_{01}^{\,2}}\right)-\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}}\ln\left(\frac{\vec{r}_{12}^{\,2}}{\tilde{\mu}^2}\right)\right]\right.\\
&\left.\times\left(\frac{3}{2}(B_{100}B_{230}+B_{200}B_{130}-B_{300}B_{210})-9B_{123}\right)+(1\leftrightarrow 3)+(2\leftrightarrow 3)\right)\\
-\frac{\alpha_s^2}{32\pi^3}\int d\vec{r}_0 &\left(B_{003}B_{012}\left[\frac{\vec{r}_{32}{}^2}{\vec{r}_{03}{}^2\vec{r}_{02}{}^2}\ln^2\left(\frac{\vec{r}_{32}{}^2\vec{r}_{10}{}^2}{\vec{r}_{13}{}^2\vec{r}_{20}{}^2}\right)-\frac{\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{02}{}^2}\ln^2\left(\frac{\vec{r}_{12}{}^2\vec{r}_{30}{}^2}{\vec{r}_{13}{}^2\vec{r}_{20}{}^2}\right)\right]\right.\\
&\left.+\,(\text{all 5 permutations}\,1\leftrightarrow 2\leftrightarrow 3)\right).
\end{aligned} \tag{4.20}$$

In the quark-diquark limit $\vec{r}_3\to\vec{r}_2$ one has

$$\begin{aligned}
&\left\{M_{12}^C\left[\left(U_0U_4{}^\dagger U_3\right)\cdot\left(U_2U_0{}^\dagger U_1\right)\cdot U_4+\left(U_1U_0{}^\dagger U_2\right)\cdot\left(U_3U_4{}^\dagger U_0\right)\cdot U_4\right]\right.\\
&\left.+\,(\text{all 5 permutations}\,1\leftrightarrow 2\leftrightarrow 3)\right\}+(0\leftrightarrow 4)\\
&\to 2\tilde{L}_{12}^C\left[tr\left(U_0{}^\dagger U_4\right)\left(tr\left(U_2{}^\dagger U_0U_4{}^\dagger U_1\right)+tr\left(U_2{}^\dagger U_1U_4{}^\dagger U_0\right)\right)\right.\\
&\left.+2tr\left(U_0{}^\dagger U_1\right)tr\left(U_2{}^\dagger U_4\right)tr\left(U_4{}^\dagger U_0\right)-(0\leftrightarrow 4)\right],
\end{aligned} \tag{4.21}$$

$$\begin{aligned}
&\left\{\tilde{L}_{12}^C\left(U_0U_4{}^\dagger U_2\right)\cdot\left(U_1U_0{}^\dagger U_4\right)\cdot U_3+(\text{all 5 permutations}\,1\leftrightarrow 2\leftrightarrow 3)\right\}+(0\leftrightarrow 4)\\
&\to 2\tilde{L}_{12}^C\left[tr\left(U_4{}^\dagger U_0\right)\left(tr\left(U_0{}^\dagger U_1U_2{}^\dagger U_4\right)+tr\left(U_0{}^\dagger U_4U_2{}^\dagger U_1\right)\right)-(0\leftrightarrow 4)\right],
\end{aligned} \tag{4.22}$$

$$\begin{aligned}
&L_{12}^C\left[\left(U_0U_4{}^\dagger U_2\right)\cdot\left(U_1U_0{}^\dagger U_4\right)\cdot U_3+tr\left(U_0U_4{}^\dagger\right)\left(U_1U_0{}^\dagger U_2\right)\cdot U_3\cdot U_4+\frac{1}{2}B_{123}\right.\\
&\left.-\frac{3}{4}[B_{144}B_{234}+B_{244}B_{134}-B_{344}B_{124}]+(\text{all 5 permutations}\,1\leftrightarrow 2\leftrightarrow 3)\right]+(0\leftrightarrow 4)\\
&\to 4L_{12}^C\left[tr\left(U_2{}^\dagger U_1\right)-3tr\left(U_0{}^\dagger U_1\right)tr\left(U_2{}^\dagger U_0\right)+tr\left(U_0{}^\dagger U_1\right)tr\left(U_2{}^\dagger U_4\right)tr\left(U_4{}^\dagger U_0\right)\right.
\end{aligned}$$

$$-tr\left(U_0{}^\dagger U_1 U_4{}^\dagger U_0 U_2{}^\dagger U_4\right)+(0\leftrightarrow 4)\Big]. \tag{4.23}$$

Therefore,

$$\langle K_{NLO}\otimes B_{122}^{conf}\rangle = -\frac{\alpha_s^2}{2\pi^4}\int d\vec{r}_0 d\vec{r}_4\ \left(\left\{\left(\tilde{L}_{12}^C+L_{12}^C\right)tr\left(U_0{}^\dagger U_1\right)tr\left(U_2{}^\dagger U_4\right)tr\left(U_4{}^\dagger U_0\right)\right.\right.$$
$$\left.\left.+L_{12}^C\left[tr\left(U_2{}^\dagger U_1\right)-3tr\left(U_0{}^\dagger U_1\right)tr\left(U_2{}^\dagger U_0\right)-tr\left(U_0{}^\dagger U_1 U_4{}^\dagger U_0 U_2{}^\dagger U_4\right)\right]\right\}+(0\leftrightarrow 4)\right)$$
$$-\frac{3\alpha_s^2}{2\pi^3}\int d\vec{r}_0\frac{11}{6}\left[\ln\left(\frac{\vec{r}_{01}^{\,2}}{\vec{r}_{02}^{\,2}}\right)\left(\frac{1}{\vec{r}_{02}^{\,2}}-\frac{1}{\vec{r}_{01}^{\,2}}\right)-\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}}\ln\left(\frac{\vec{r}_{12}^{\,2}}{\tilde{\mu}^2}\right)\right]$$
$$\times\left(tr\left(U_0{}^\dagger U_1\right)tr\left(U_2{}^\dagger U_0\right)-3tr\left(U_2{}^\dagger U_1\right)\right). \tag{4.24}$$

This is twice the gluon part of the BK kernel (see (67) in [6]).

5 Linearization

In the 3-gluon approximation

$$B_{003}B_{012}\overset{3g}{=}6B_{003}+6B_{012}-36. \tag{5.1}$$

We use the following identity to linearize the color structures in (4.20).

$$(U_0U_4^\dagger U_2)\cdot(U_1U_0^\dagger U_4)\cdot U_3+(1\leftrightarrow 2,0\leftrightarrow 4)=$$
$$=(U_0U_4^\dagger-E)(U_2-U_4)\cdot(U_1-U_0)U_0^\dagger U_4\cdot U_3+U_4U_0^\dagger(U_1-U_0)\cdot(U_2-U_4)(U_4^\dagger U_0-E)\cdot U_3$$
$$+U_0\cdot(U_1U_0^\dagger U_4+U_4U_0^\dagger U_1)\cdot U_3+(U_0U_4^\dagger U_2+U_2U_4^\dagger U_0)\cdot U_4\cdot U_3$$
$$+(U_2-U_4)\cdot(U_1-U_0)(U_0^\dagger U_4-E)\cdot U_3+(U_4U_0^\dagger-E)(U_1-U_0)\cdot(U_2-U_4)\cdot U_3$$
$$+2(U_2-U_4)\cdot(U_1-U_0)\cdot U_3-2U_0\cdot U_4\cdot U_3. \tag{5.2}$$

Here E is the identity matrix. In the 3-gluon approximation it reads

$$(U_0U_4^\dagger U_2)\cdot(U_1U_0^\dagger U_4)\cdot U_3+(1\leftrightarrow 2,0\leftrightarrow 4)$$
$$\overset{3g}{=}(U_0-U_4)(U_2-U_4)\cdot(U_1-U_0)\cdot E+(U_1-U_0)\cdot(U_2-U_4)(U_0-U_4)\cdot E$$
$$+U_0\cdot(U_1U_0^\dagger U_4+U_4U_0^\dagger U_1)\cdot U_3+(U_0U_4^\dagger U_2+U_2U_4^\dagger U_0)\cdot U_4\cdot U_3$$
$$+(U_2-U_4)\cdot(U_1-U_0)(U_4-U_0)\cdot E+(U_4-U_0)(U_1-U_0)\cdot(U_2-U_4)\cdot E$$
$$+2(U_2-U_4)\cdot(U_1-U_0)\cdot U_3-2U_0\cdot U_4\cdot U_3. \tag{5.3}$$

Using identity (A.3) and the fact that in the 3-gluon approximation

$$((U_0-U_4)(U_2-U_4)+(U_2-U_4)(U_0-U_4))\cdot(U_1-U_0)\cdot E$$
$$\overset{3g}{=}-(U_2-U_4)\cdot(U_0-U_4)\cdot(U_1-U_0), \tag{5.4}$$

we get

$$
\begin{aligned}
&(U_0U_4^\dagger U_2)\cdot(U_1U_0^\dagger U_4)\cdot U_3+(1\leftrightarrow 2,0\leftrightarrow 4)\\
&\overset{3g}{=}-B_{134}+\frac{1}{2}(B_{100}B_{340}+B_{400}B_{130}-B_{300}B_{140})\\
&\quad -B_{023}+\frac{1}{2}(B_{044}B_{234}+B_{244}B_{034}-B_{344}B_{024})\\
&\quad +2(U_2-U_4)\cdot(U_1-U_0)\cdot U_3-2U_0\cdot U_4\cdot U_3\\
&=B_{123}-3B_{134}+\frac{1}{2}(B_{100}B_{340}+B_{400}B_{130}-B_{300}B_{140})+(1\leftrightarrow 2,0\leftrightarrow 4)\\
&\overset{3g}{=}B_{123}+3(B_{100}+B_{340}+B_{400}+B_{130}-B_{300}-B_{140}-B_{134}-6)+(1\leftrightarrow 2,0\leftrightarrow 4).
\end{aligned}
\tag{5.5}
$$

As a result the coefficient of $\tilde{L}_{12}^C$ in (4.20) reads

$$
\begin{aligned}
&(\left(U_0{U_4}^\dagger U_2\right)\cdot\left(U_1{U_0}^\dagger U_4\right)\cdot U_3+(1\leftrightarrow 2,0\leftrightarrow 4))-(0\leftrightarrow 4)\\
&\overset{3g}{=}(3B_{001}+6B_{130}-(1\leftrightarrow 2))-(0\leftrightarrow 4).
\end{aligned}
\tag{5.6}
$$

Using integrals (114) and (125) from [6],

$$
\int d\vec{r}_4\tilde{L}_{12}=\frac{\pi^2}{2}\zeta\left(3\right)\left(\delta\left(\vec{r}_{10}\right)-\delta\left(\vec{r}_{20}\right)\right),
\tag{5.7}
$$

and

$$
\begin{aligned}
&\int d\vec{r}_4\left[\frac{{\vec{r}_{12}}^2}{4{\vec{r}_{01}}^2{\vec{r}_{04}}^2{\vec{r}_{24}}^2}\ln\left(\frac{{\vec{r}_{02}}^2{\vec{r}_{14}}^2}{{\vec{r}_{04}}^2{\vec{r}_{12}}^2}\right)-\frac{{\vec{r}_{12}}^2}{4{\vec{r}_{02}}^2{\vec{r}_{04}}^2{\vec{r}_{14}}^2}\ln\left(\frac{{\vec{r}_{01}}^2{\vec{r}_{24}}^2}{{\vec{r}_{04}}^2{\vec{r}_{12}}^2}\right)\right]\\
&=\pi^2\zeta\left(3\right)\left(\delta\left(\vec{r}_{10}\right)-\delta\left(\vec{r}_{20}\right)\right),
\end{aligned}
\tag{5.8}
$$

one has

$$
\int d\vec{r}_4\tilde{L}_{12}^C=\frac{3}{2}\pi^2\zeta\left(3\right)\left(\delta\left(\vec{r}_{10}\right)-\delta\left(\vec{r}_{20}\right)\right),
\tag{5.9}
$$

and

$$
\begin{aligned}
&-\frac{\alpha_s^2}{8\pi^4}\int d\vec{r}_0d\vec{r}_4\ \left(\left\{\tilde{L}_{12}^C\left(U_0{U_4}^\dagger U_2\right)\cdot\left(U_1{U_0}^\dagger U_4\right)\cdot U_3\right.\right.\\
&\quad\left.\left.+\left(\text{all 5 permutations}\,1\leftrightarrow 2\leftrightarrow 3\right)\right\}+(0\leftrightarrow 4)\right)\\
&\overset{3g}{=}-\frac{\alpha_s^2}{8\pi^4}\int d\vec{r}_0(3B_{001}+6B_{130}-(1\leftrightarrow 2))3\pi^2\zeta\left(3\right)\left(\delta\left(\vec{r}_{10}\right)-\delta\left(\vec{r}_{20}\right)\right)+(1\leftrightarrow 3)+(2\leftrightarrow 3)\\
&=-\frac{9\alpha_s^2}{8\pi^2}\zeta\left(3\right)(36+B_{131}+B_{133}+B_{121}+B_{212}+B_{232}+B_{233}-12B_{231})).
\end{aligned}
\tag{5.10}
$$

The second structure reads

$$
\begin{aligned}
&\left(U_1{U_0}^\dagger U_2+U_2{U_0}^\dagger U_1\right)\cdot U_3\cdot U_4\\
&=(U_1-U_0){U_0}^\dagger(U_2-U_0))\cdot U_3\cdot(U_4-U_0)+(U_2-U_0){U_0}^\dagger(U_1-U_0))\cdot U_3\cdot(U_4-U_0)\\
&\quad+2(U_2+U_1-U_0)\cdot U_3\cdot(U_4-U_0)+\left(U_1{U_0}^\dagger U_2+U_2{U_0}^\dagger U_1\right)\cdot U_3\cdot U_0.
\end{aligned}
\tag{5.11}
$$

Again, applying identity (A.3) and equality (5.4) one gets in the 3-gluon approximation

$$\left(U_1{U_0}^{\dagger}U_2+U_2{U_0}^{\dagger}U_1\right)\cdot U_3\cdot U_4\overset{3g}{=}-(U_1-U_0)\cdot(U_2-U_0)\cdot(U_4-U_0)$$

$$+\,2(U_2+U_1-U_0)\cdot U_3\cdot(U_4-U_0)-B_{123}+\frac{1}{2}\left(B_{100}B_{230}+B_{200}B_{130}-B_{300}B_{120}\right). \quad (5.12)$$

Finally, the coefficient of L_{12}^C in (4.20) reads

$$\left[\left(U_0{U_4}^{\dagger}U_2\right)\cdot\left(U_1{U_0}^{\dagger}U_4\right)\cdot U_3+tr\left(U_0{U_4}^{\dagger}\right)\left(U_1{U_0}^{\dagger}U_2\right)\cdot U_3\cdot U_4\right.$$
$$\left.-\frac{3}{4}[B_{144}B_{234}+B_{244}B_{134}-B_{344}B_{124}]+\frac{1}{2}B_{123}+(1\leftrightarrow 2)\right]+(0\leftrightarrow 4)$$
$$\overset{3g}{=}9(B_{044}+B_{004}-12). \quad (5.13)$$

Therefore,

$$-\frac{\alpha_s^2}{8\pi^4}\int d\vec{r}_0 d\vec{r}_4\ \left(\left\{L_{12}^C\left[\left(U_0{U_4}^{\dagger}U_2\right)\cdot\left(U_1{U_0}^{\dagger}U_4\right)\cdot U_3+tr\left(U_0{U_4}^{\dagger}\right)\left(U_1{U_0}^{\dagger}U_2\right)\cdot U_3\cdot U_4\right.\right.\right.$$
$$\left.-\frac{3}{4}[B_{144}B_{234}+B_{244}B_{134}-B_{344}B_{124}]+\frac{1}{2}B_{123}\right]$$
$$\left.\left.+\,(\text{all 5 permutations}\,1\leftrightarrow 2\leftrightarrow 3)\right\}+(0\leftrightarrow 4)\right)$$
$$\overset{3g}{=}-\frac{9\alpha_s^2}{8\pi^4}\int d\vec{r}_0 d\vec{r}_4(L_{12}^C+L_{13}^C+L_{23}^C)(B_{044}+B_{004}-12)\,. \quad (5.14)$$

The third structure reads

$$(U_2U_0^{\dagger}U_1)\cdot U_4\cdot(U_0U_4^{\dagger}U_3)+(U_1U_0^{\dagger}U_2)\cdot U_4\cdot(U_3U_4^{\dagger}U_0)$$
$$=U_1\cdot U_4\cdot(U_3U_4^{\dagger}U_0+U_0U_4^{\dagger}U_3)+(U_1U_0^{\dagger}U_2+U_2U_0^{\dagger}U_1)\cdot U_4\cdot U_0$$
$$+(U_2-U_0)\cdot U_4\cdot((U_0-U_4)U_4^{\dagger}(U_3-U_4)+(U_3-U_4)U_4^{\dagger}(U_0-U_4))$$
$$+((U_1-U_0)U_0^{\dagger}(U_2-U_0))\cdot U_4\cdot((U_3-U_4)U_4^{\dagger}U_0)$$
$$+((U_2-U_0)U_0^{\dagger}(U_1-U_0))\cdot U_4\cdot(U_0U_4^{\dagger}(U_3-U_4))$$
$$+\,2(U_2-U_0)\cdot U_4\cdot(U_3-U_4)-2U_1\cdot U_4\cdot U_0. \quad (5.15)$$

Using (A.3) and (5.4),

$$(U_2U_0^{\dagger}U_1)\cdot U_4\cdot(U_0U_4^{\dagger}U_3)+(U_1U_0^{\dagger}U_2)\cdot U_4\cdot(U_3U_4^{\dagger}U_0)$$
$$\overset{3g}{=}-B_{013}+\frac{1}{2}(B_{344}B_{014}+B_{044}B_{134}-B_{144}B_{034})-B_{124}+\frac{1}{2}(B_{100}B_{240}+B_{200}B_{140}-B_{004}B_{012})$$
$$-(U_2-U_0)\cdot(U_1-3U_4)\cdot(U_3-U_4)-2U_1\cdot U_4\cdot U_0$$
$$\overset{3g}{=}3(B_{010}-B_{441}+B_{020}-B_{442}-B_{040}+2B_{440}-B_{120}+B_{140}$$
$$+\,B_{341}+B_{240}-2B_{340}+B_{342}+B_{443})-B_{231}-36. \quad (5.16)$$

As a result

$$-\frac{\alpha_s^2}{8\pi^4}\int d\vec r_0 d\vec r_4 \;\; \Big(\Big\{M_{12}^C\Big[\Big(U_0U_4{}^\dagger U_3\Big)\cdot\Big(U_2U_0{}^\dagger U_1\Big)\cdot U_4+\Big(U_1U_0{}^\dagger U_2\Big)\cdot\Big(U_3U_4{}^\dagger U_0\Big)\cdot U_4\Big] + (\text{all 5 permutations}\, 1\leftrightarrow 2\leftrightarrow 3)\Big\}+(0\leftrightarrow 4)\Big)$$

$$\stackrel{\text{3g}}{=} -\frac{3\alpha_s^2}{32\pi^4}\int d\vec r_0 d\vec r_4 \;\; \Big(\frac{3}{2}F_0(B_{040}-B_{044})+\Big\{\frac{3}{2}F_{140}B_{140}+F_{100}B_{100}+F_{230}B_{230} + (0\leftrightarrow 4)\Big\}+(\text{all 5 permutations}\, 1\leftrightarrow 2\leftrightarrow 3)\Big). \tag{5.17}$$

Here

$$F_0=\frac{\vec r_{12}^{\,2}}{2\vec r_{14}^{\,2}\vec r_{24}^{\,2}}\left(\frac{\vec r_{24}^{\,2}}{\vec r_{02}^{\,2}\vec r_{04}^{\,2}}\ln\left(\frac{\vec r_{01}^{\,2}\vec r_{02}^{\,2}\vec r_{34}^{\,4}}{\vec r_{14}^{\,2}\vec r_{24}^{\,2}\vec r_{03}^{\,4}}\right)-\frac{\vec r_{13}^{\,2}}{\vec r_{01}^{\,2}\vec r_{03}^{\,2}}\ln\left(\frac{\vec r_{01}^{\,2}\vec r_{13}^{\,2}\vec r_{24}^{\,2}}{\vec r_{03}^{\,2}\vec r_{12}^{\,2}\vec r_{14}^{\,2}}\right) +\frac{2\vec r_{34}^{\,2}}{\vec r_{03}^{\,2}\vec r_{04}^{\,2}}\ln\left(\frac{\vec r_{01}^{\,2}\vec r_{02}^{\,2}\vec r_{34}^{\,2}}{\vec r_{03}^{\,2}\vec r_{04}^{\,2}\vec r_{12}^{\,2}}\right)\right)-(0\leftrightarrow 4). \tag{5.18}$$

$$F_{140}=\frac{\vec r_{12}^{\,2}}{\vec r_{02}^{\,2}\vec r_{04}^{\,2}\vec r_{14}^{\,2}}\ln\left(\frac{\vec r_{02}^{\,2}\vec r_{04}^{\,2}\vec r_{12}^{\,2}\vec r_{34}^{\,4}}{\vec r_{03}^{\,4}\vec r_{14}^{\,2}\vec r_{24}^{\,4}}\right) -\frac{\vec r_{01}^{\,2}\vec r_{23}^{\,2}}{\vec r_{02}^{\,2}\vec r_{03}^{\,2}\vec r_{04}^{\,2}\vec r_{14}^{\,2}}\ln\left(\frac{\vec r_{01}^{\,2}\vec r_{24}^{\,2}\vec r_{34}^{\,2}}{\vec r_{04}^{\,2}\vec r_{14}^{\,2}\vec r_{23}^{\,2}}\right)-\frac{\vec r_{23}^{\,2}\vec r_{12}^{\,2}}{\vec r_{02}^{\,2}\vec r_{03}^{\,2}\vec r_{14}^{\,2}\vec r_{24}^{\,2}}\ln\left(\frac{\vec r_{02}^{\,2}\vec r_{14}^{\,2}\vec r_{23}^{\,2}}{\vec r_{03}^{\,2}\vec r_{12}^{\,2}\vec r_{24}^{\,2}}\right) +\frac{\vec r_{23}^{\,2}}{\vec r_{03}^{\,2}\vec r_{04}^{\,2}\vec r_{24}^{\,2}}\ln\left(\frac{\vec r_{02}^{\,2}\vec r_{34}^{\,2}}{\vec r_{04}^{\,2}\vec r_{23}^{\,2}}\right)+\frac{\vec r_{02}^{\,2}\vec r_{13}^{\,2}}{\vec r_{01}^{\,2}\vec r_{03}^{\,2}\vec r_{04}^{\,2}\vec r_{24}^{\,2}}\ln\left(\frac{\vec r_{01}^{\,2}\vec r_{02}^{\,2}\vec r_{34}^{\,4}}{\vec r_{03}^{\,2}\vec r_{04}^{\,2}\vec r_{13}^{\,2}\vec r_{24}^{\,2}}\right). \tag{5.19}$$

$$F_{100}=\frac{\vec r_{23}^{\,2}}{2\vec r_{03}^{\,2}\vec r_{04}^{\,2}\vec r_{24}^{\,2}}\ln\left(\frac{\vec r_{01}^{\,8}\vec r_{04}^{\,2}\vec r_{23}^{\,2}\vec r_{24}^{\,4}\vec r_{34}^{\,2}}{\vec r_{02}^{\,6}\vec r_{03}^{\,4}\vec r_{14}^{\,8}}\right)-\frac{\vec r_{02}^{\,2}\vec r_{13}^{\,2}}{\vec r_{01}^{\,2}\vec r_{03}^{\,2}\vec r_{04}^{\,2}\vec r_{24}^{\,2}}\ln\left(\frac{\vec r_{01}^{\,2}\vec r_{04}^{\,2}\vec r_{13}^{\,2}\vec r_{24}^{\,2}}{\vec r_{02}^{\,2}\vec r_{03}^{\,2}\vec r_{14}^{\,4}}\right) -\frac{\vec r_{34}^{\,2}\vec r_{12}^{\,2}}{2\vec r_{03}^{\,2}\vec r_{04}^{\,2}\vec r_{14}^{\,2}\vec r_{24}^{\,2}}\ln\left(\frac{\vec r_{01}^{\,8}\vec r_{02}^{\,4}\vec r_{24}^{\,2}\vec r_{34}^{\,6}}{\vec r_{03}^{\,6}\vec r_{04}^{\,6}\vec r_{12}^{\,6}\vec r_{14}^{\,2}}\right)-\frac{\vec r_{12}^{\,2}}{2\vec r_{02}^{\,2}\vec r_{04}^{\,2}\vec r_{14}^{\,2}}\ln\left(\frac{\vec r_{01}^{\,4}\vec r_{02}^{\,2}\vec r_{34}^{\,4}}{\vec r_{03}^{\,4}\vec r_{04}^{\,2}\vec r_{12}^{\,2}\vec r_{14}^{\,2}}\right) +\frac{\vec r_{23}^{\,2}\vec r_{12}^{\,2}}{2\vec r_{02}^{\,2}\vec r_{03}^{\,2}\vec r_{14}^{\,2}\vec r_{24}^{\,2}}\ln\left(\frac{\vec r_{02}^{\,2}\vec r_{14}^{\,2}\vec r_{23}^{\,2}}{\vec r_{03}^{\,2}\vec r_{12}^{\,2}\vec r_{24}^{\,2}}\right)+\frac{\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec r_{04}^{\,2}\vec r_{24}^{\,2}}\ln\left(\frac{\vec r_{02}^{\,2}\vec r_{14}^{\,2}}{\vec r_{04}^{\,2}\vec r_{12}^{\,2}}\right) +\frac{\vec r_{13}^{\,2}\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec r_{03}^{\,2}\vec r_{14}^{\,2}\vec r_{24}^{\,2}}\ln\left(\frac{\vec r_{01}^{\,2}\vec r_{13}^{\,2}\vec r_{24}^{\,2}}{\vec r_{03}^{\,2}\vec r_{12}^{\,2}\vec r_{14}^{\,2}}\right)+\frac{\vec r_{01}^{\,2}\vec r_{23}^{\,2}}{2\vec r_{02}^{\,2}\vec r_{03}^{\,2}\vec r_{04}^{\,2}\vec r_{14}^{\,2}}\ln\left(\frac{\vec r_{01}^{\,2}\vec r_{24}^{\,2}\vec r_{34}^{\,2}}{\vec r_{04}^{\,2}\vec r_{14}^{\,2}\vec r_{23}^{\,2}}\right). \tag{5.20}$$

$$F_{230}=\frac{\vec r_{02}^{\,2}\vec r_{13}^{\,2}}{2\vec r_{01}^{\,2}\vec r_{03}^{\,2}\vec r_{04}^{\,2}\vec r_{24}^{\,2}}\ln\left(\frac{\vec r_{01}^{\,2}\vec r_{04}^{\,2}\vec r_{13}^{\,2}\vec r_{24}^{\,2}}{\vec r_{02}^{\,2}\vec r_{03}^{\,2}\vec r_{14}^{\,4}}\right)-\frac{\vec r_{23}^{\,2}}{2\vec r_{03}^{\,2}\vec r_{04}^{\,2}\vec r_{24}^{\,2}}\ln\left(\frac{\vec r_{01}^{\,4}\vec r_{04}^{\,4}\vec r_{23}^{\,4}\vec r_{24}^{\,2}}{\vec r_{02}^{\,6}\vec r_{03}^{\,2}\vec r_{14}^{\,4}\vec r_{34}^{\,2}}\right) +\frac{\vec r_{34}^{\,2}\vec r_{12}^{\,2}}{2\vec r_{03}^{\,2}\vec r_{04}^{\,2}\vec r_{14}^{\,2}\vec r_{24}^{\,2}}\ln\left(\frac{\vec r_{01}^{\,4}\vec r_{02}^{\,8}\vec r_{14}^{\,2}\vec r_{34}^{\,6}}{\vec r_{03}^{\,6}\vec r_{04}^{\,6}\vec r_{12}^{\,6}\vec r_{24}^{\,2}}\right)+\frac{\vec r_{12}^{\,2}}{2\vec r_{02}^{\,2}\vec r_{04}^{\,2}\vec r_{14}^{\,2}}\ln\left(\frac{\vec r_{01}^{\,2}\vec r_{02}^{\,4}\vec r_{04}^{\,2}\vec r_{12}^{\,2}\vec r_{34}^{\,8}}{\vec r_{03}^{\,8}\vec r_{14}^{\,4}\vec r_{24}^{\,6}}\right) -\frac{\vec r_{23}^{\,2}\vec r_{12}^{\,2}}{\vec r_{02}^{\,2}\vec r_{03}^{\,2}\vec r_{14}^{\,2}\vec r_{24}^{\,2}}\ln\left(\frac{\vec r_{02}^{\,2}\vec r_{14}^{\,2}\vec r_{23}^{\,2}}{\vec r_{03}^{\,2}\vec r_{12}^{\,2}\vec r_{24}^{\,2}}\right)-\frac{\vec r_{12}^{\,2}}{2\vec r_{01}^{\,2}\vec r_{04}^{\,2}\vec r_{24}^{\,2}}\ln\left(\frac{\vec r_{02}^{\,2}\vec r_{14}^{\,2}}{\vec r_{04}^{\,2}\vec r_{12}^{\,2}}\right) -\frac{\vec r_{13}^{\,2}\vec r_{12}^{\,2}}{2\vec r_{01}^{\,2}\vec r_{03}^{\,2}\vec r_{14}^{\,2}\vec r_{24}^{\,2}}\ln\left(\frac{\vec r_{01}^{\,2}\vec r_{13}^{\,2}\vec r_{24}^{\,2}}{\vec r_{03}^{\,2}\vec r_{12}^{\,2}\vec r_{14}^{\,2}}\right)-\frac{\vec r_{01}^{\,2}\vec r_{23}^{\,2}}{\vec r_{02}^{\,2}\vec r_{03}^{\,2}\vec r_{04}^{\,2}\vec r_{14}^{\,2}}\ln\left(\frac{\vec r_{01}^{\,2}\vec r_{24}^{\,2}\vec r_{34}^{\,2}}{\vec r_{04}^{\,2}\vec r_{14}^{\,2}\vec r_{23}^{\,2}}\right). \tag{5.21}$$

One can integrate F_{100} and F_{230} w.r.t. $\vec{r}_4$. The integrals are given in appendix C (C.23) and (C.33). Putting things together we have for the linearized kernel

$$\langle K_{NLO} \otimes B_{123}^{conf} \rangle \overset{3g}{=} -\frac{9\alpha_s^2}{8\pi^4}\int d\vec{r}_0 d\vec{r}_4 (L_{12}^C + L_{13}^C + L_{23}^C)(B_{044} + B_{004} - 12)$$
$$+\frac{27\alpha_s^2}{4\pi^2}\zeta(3)(3 - \delta_{23} - \delta_{13} - \delta_{21})(B_{123} - 6)$$
$$-\frac{9\alpha_s^2}{64\pi^4}\int d\vec{r}_0 d\vec{r}_4 \ (F_0(B_{040} - B_{044}) + \{F_{140} + (0 \leftrightarrow 4)\} B_{140} + (\text{all 5 perm.}\, 1 \leftrightarrow 2 \leftrightarrow 3))$$
$$-\frac{9\alpha_s^2}{64\pi^3}\int d\vec{r}_0 \left(\tilde{F}_{100}B_{100} + \tilde{F}_{230}B_{230} + (1 \leftrightarrow 3) + (1 \leftrightarrow 2)\right)$$
$$-\frac{9\alpha_s^2}{16\pi^3}\int d\vec{r}_0 \left(\frac{11}{3}\left[\ln\left(\frac{\vec{r}_{01}^{\,2}}{\vec{r}_{02}^{\,2}}\right)\left(\frac{1}{\vec{r}_{02}^{\,2}} - \frac{1}{\vec{r}_{01}^{\,2}}\right) - \frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}}\ln\left(\frac{\vec{r}_{12}^{\,2}}{\tilde{\mu}^2}\right)\right]\right.$$
$$\left.\times (B_{100} + B_{230} + B_{200} + B_{130} - B_{300} - B_{210} - B_{123} - 6) + (1 \leftrightarrow 3) + (2 \leftrightarrow 3)\right). \quad (5.22)$$

Here $\delta_{ij} = 1$, if $\vec{r}_i = \vec{r}_j$ and $\delta_{ij} = 0$ otherwise; F_0 and F_{140} are defined in (5.18) and (5.19); L_{12}^C is defined in (4.15)

$$L_{12}^C = \frac{\vec{r}_{12}{}^2}{4\vec{r}_{01}{}^2\vec{r}_{04}{}^2\vec{r}_{24}{}^2}\ln\left(\frac{\vec{r}_{02}{}^2\vec{r}_{14}{}^2}{\vec{r}_{04}{}^2\vec{r}_{12}{}^2}\right) + \frac{\vec{r}_{12}{}^2}{4\vec{r}_{02}{}^2\vec{r}_{04}{}^2\vec{r}_{14}{}^2}\ln\left(\frac{\vec{r}_{01}{}^2\vec{r}_{24}{}^2}{\vec{r}_{04}{}^2\vec{r}_{12}{}^2}\right)$$
$$+\left[\frac{1}{\vec{r}_{01}{}^2\vec{r}_{24}{}^2 - \vec{r}_{02}{}^2\vec{r}_{14}{}^2}\left(-\frac{\vec{r}_{12}{}^4}{8}\left(\frac{1}{\vec{r}_{01}{}^2\vec{r}_{24}{}^2} + \frac{1}{\vec{r}_{02}{}^2\vec{r}_{14}{}^2}\right) + \frac{\vec{r}_{12}{}^2}{\vec{r}_{04}{}^2} - \frac{\vec{r}_{02}{}^2\vec{r}_{14}{}^2 + \vec{r}_{01}{}^2\vec{r}_{24}{}^2}{4\vec{r}_{04}{}^4}\right)\right.$$
$$\left.+\frac{\vec{r}_{12}{}^2}{8\vec{r}_{04}{}^2}\left(\frac{1}{\vec{r}_{02}{}^2\vec{r}_{14}{}^2} - \frac{1}{\vec{r}_{01}{}^2\vec{r}_{24}{}^2}\right)\right]\ln\left(\frac{\vec{r}_{01}{}^2\vec{r}_{24}{}^2}{\vec{r}_{14}{}^2\vec{r}_{02}{}^2}\right) + \frac{1}{2\vec{r}_{04}{}^4} \quad (5.23)$$

and

$$\tilde{F}_{100} = \left(\frac{\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{02}{}^2} - \frac{\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{03}{}^2} - \frac{2\vec{r}_{23}{}^2}{\vec{r}_{02}{}^2\vec{r}_{03}{}^2}\right)\ln^2\left(\frac{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}\right) + \frac{\vec{r}_{23}{}^2}{2\vec{r}_{02}{}^2\vec{r}_{03}{}^2}\ln^2\left(\frac{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}\right)$$
$$+\tilde{S}_{123}I\left(\frac{\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{02}{}^2}, \frac{\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{03}{}^2}, \frac{\vec{r}_{23}{}^2}{\vec{r}_{02}{}^2\vec{r}_{03}{}^2}\right) + (2 \leftrightarrow 3), \quad (5.24)$$

$$\tilde{F}_{230} = \left(\frac{2\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{02}{}^2} - \frac{\vec{r}_{23}{}^2}{2\vec{r}_{02}{}^2\vec{r}_{03}{}^2}\right)\ln^2\left(\frac{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}\right) + \left(\frac{\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{03}{}^2} - \frac{\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{02}{}^2}\right)\ln^2\left(\frac{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}\right)$$
$$-\tilde{S}_{123}I\left(\frac{\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{02}{}^2}, \frac{\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{03}{}^2}, \frac{\vec{r}_{23}{}^2}{\vec{r}_{02}{}^2\vec{r}_{03}{}^2}\right) + (2 \leftrightarrow 3). \quad (5.25)$$

The functions $\tilde{S}_{123}$ and I are defined in appendix C (C.16) and (C.12). If we consider the dipole limit $\vec{r}_3 = \vec{r}_2$ and take into account that in this limit

$$\tilde{F}_{230}|_{\vec{r}_3=\vec{r}_2} = \tilde{F}_{130}|_{\vec{r}_3=\vec{r}_2} = \tilde{F}_{210}|_{\vec{r}_3=\vec{r}_2} = \tilde{F}_{100}|_{\vec{r}_3=\vec{r}_2} = \tilde{F}_{200}|_{\vec{r}_3=\vec{r}_2} = \tilde{F}_{300}|_{\vec{r}_3=\vec{r}_2} = 0, \quad (5.26)$$

$$F_0 + (\text{all 5 permutations}\, 1 \leftrightarrow 2 \leftrightarrow 3)|_{\vec{r}_3=\vec{r}_2} = -16\tilde{L}_{12}^C, \quad (5.27)$$
$$(F_{140} + (0 \leftrightarrow 4)) + (2 \leftrightarrow 3)|_{\vec{r}_3=\vec{r}_2} = 0,$$

$$(F_{240} + (0 \leftrightarrow 4)) + (1 \leftrightarrow 3)|_{\vec{r}_3=\vec{r}_2} = (F_{340} + (0 \leftrightarrow 4)) + (2 \leftrightarrow 1)|_{\vec{r}_3=\vec{r}_2} = 0, \tag{5.28}$$

we have the linearized BK kernel [6], whose C-even part is the BFKL kernel [15]

$$\begin{aligned}\langle K_{NLO} \otimes B_{122}^{conf}\rangle \overset{3g}{=} -\frac{9\alpha_s^2}{4\pi^4}\int d\vec{r}_0 d\vec{r}_4 L_{12}^C (B_{044} + B_{004} - 12) + \frac{27\alpha_s^2}{2\pi^2}\zeta(3)(B_{122} - 6) \\ -\frac{9\alpha_s^2}{4\pi^4}\int d\vec{r}_0 d\vec{r}_4\ \tilde{L}_{12}^C (B_{044} - B_{040}) \\ -\frac{9\alpha_s^2}{8\pi^3}\frac{11}{3}\int d\vec{r}_0 \left[\ln\left(\frac{\vec{r}_{01}^{\,2}}{\vec{r}_{02}^{\,2}}\right)\left(\frac{1}{\vec{r}_{02}^{\,2}} - \frac{1}{\vec{r}_{01}^{\,2}}\right) - \frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}}\ln\left(\frac{\vec{r}_{12}^{\,2}}{\tilde{\mu}^2}\right)\right](B_{100} + B_{220} - B_{122} - 6)\,.\end{aligned} \tag{5.29}$$

As in [10] to separate the C-even and C-odd contributions we introduce C-even (pomeron) and C-odd (odderon) Green functions

$$B_{123}^{+} = B_{123} + B_{\bar{1}\bar{2}\bar{3}} - 12, \tag{5.30}$$

and

$$B_{123}^{-} = B_{123} - B_{\bar{1}\bar{2}\bar{3}}, \tag{5.31}$$

where $B_{\bar{1}\bar{2}\bar{3}}$ is the 3-antiquark Wilson loop operator

$$B_{\bar{1}\bar{2}\bar{3}} = U_1^{\dagger} \cdot U_2^{\dagger} \cdot U_3^{\dagger}. \tag{5.32}$$

$$\begin{aligned}\langle K_{NLO} \otimes B_{123}^{+conf}\rangle \overset{3g}{=} -\frac{9\alpha_s^2}{4\pi^4}\int d\vec{r}_0 d\vec{r}_4 (L_{12}^C + L_{13}^C + L_{23}^C) B_{044}^{+} + \frac{27\alpha_s^2}{4\pi^2}\zeta(3)(3 - \delta_{23} - \delta_{13} - \delta_{21}) B_{123}^{+} \\ -\frac{9\alpha_s^2}{64\pi^4}\int d\vec{r}_0 d\vec{r}_4\ \left(\{F_{140} + (0 \leftrightarrow 4)\} B_{140}^{+} + (\text{all 5 permutations}\, 1 \leftrightarrow 2 \leftrightarrow 3)\right) \\ -\frac{9\alpha_s^2}{64\pi^3}\int d\vec{r}_0 \left(\tilde{F}_{100} B_{100}^{+} + \tilde{F}_{230} B_{230}^{+} + (1 \leftrightarrow 3) + (1 \leftrightarrow 2)\right) \\ -\frac{9\alpha_s^2}{16\pi^3}\int d\vec{r}_0 \left(\frac{11}{3}\left[\ln\left(\frac{\vec{r}_{01}^{\,2}}{\vec{r}_{02}^{\,2}}\right)\left(\frac{1}{\vec{r}_{02}^{\,2}} - \frac{1}{\vec{r}_{01}^{\,2}}\right) - \frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}}\ln\left(\frac{\vec{r}_{12}^{\,2}}{\tilde{\mu}^2}\right)\right]\right. \\ \left.\times\left(B_{100}^{+} + B_{230}^{+} + B_{200}^{+} + B_{130}^{+} - B_{300}^{+} - B_{210}^{+} - B_{123}^{+}\right) + (1 \leftrightarrow 3) + (2 \leftrightarrow 3)\right).\end{aligned} \tag{5.33}$$

In the 3-gluon approximation [10]

$$B_{123}^{+} \overset{3g}{=} \frac{1}{2}(B_{133}^{+} + B_{211}^{+} + B_{322}^{+}). \tag{5.34}$$

Therefore for model (4.1)

$$B_{123}^{+conf} \overset{3g}{=} \frac{1}{2}(B_{133}^{+conf} + B_{211}^{+conf} + B_{322}^{+conf}) \tag{5.35}$$

and

$$\langle K_{NLO} \otimes B_{123}^{+conf}\rangle \overset{3g}{=} \frac{1}{2}\langle K_{NLO} \otimes (B_{133}^{+conf} + B_{211}^{+conf} + B_{322}^{+conf})\rangle. \tag{5.36}$$

This equality imposes the following constraints

$$0 = \ \{F_{140} + (0 \leftrightarrow 4)\} + (\text{all 5 permutations}\, 1 \leftrightarrow 2 \leftrightarrow 3), \tag{5.37}$$

$$0=\int d\vec{r}_0\tilde{F}_{230}, \tag{5.38}$$

$$0=\int\frac{d\vec{r}_4}{\pi}\ \left(\{F_{140}+(0\leftrightarrow 4)\}+(2\leftrightarrow 3)\right)+\tilde{F}_{100}+\frac{1}{2}\tilde{F}_{230}|_{1\leftrightarrow 3}+\frac{1}{2}\tilde{F}_{230}|_{1\leftrightarrow 2}. \tag{5.39}$$

Constraint (5.37) follows from definition of F_{140} (5.19) directly. Constraint (5.38) holds since thanks to conformal invariance

$$\int d\vec{r}_0\tilde{F}_{230}=\int d\vec{r}_0\tilde{F}_{230}|_{2=3}=0. \tag{5.40}$$

Using (5.24) and (5.25) one can rewrite constraint (5.39) as

$$\int\frac{d\vec{r}_4}{\pi}\ \left(\{F_{140}+(0\leftrightarrow 4)\}+(2\leftrightarrow 3)\right)=\frac{\vec{r}_{23}{}^2}{2\vec{r}_{02}{}^2\vec{r}_{03}{}^2}\left(\ln^2\left(\frac{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}\right)+\ln^2\left(\frac{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}\right)\right)$$
$$-\frac{1}{2}\left(\frac{\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{02}{}^2}+\frac{\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{03}{}^2}\right)\ln^2\left(\frac{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}\right). \tag{5.41}$$

The calculation of the integral and proof of this identity is given in appendix C.

$$\langle K_{NLO}\otimes B_{123}^{-conf}\rangle \overset{3g}{=} \frac{27\alpha_s^2}{4\pi^2}\zeta(3)(3-\delta_{23}-\delta_{13}-\delta_{21})B^{-}_{123}$$
$$-\frac{9\alpha_s^2}{64\pi^4}\int d\vec{r}_0 d\vec{r}_4\ \left(2F_0B^{-}_{040}+\{F_{140}+(0\leftrightarrow 4)\}\,B^{-}_{140}+(\text{all 5 permutations}\,1\leftrightarrow 2\leftrightarrow 3)\right)$$
$$-\frac{9\alpha_s^2}{64\pi^3}\int d\vec{r}_0\left(\tilde{F}_{100}B^{-}_{100}+\tilde{F}_{230}B^{-}_{230}+(1\leftrightarrow 3)+(1\leftrightarrow 2)\right)$$
$$-\frac{9\alpha_s^2}{16\pi^3}\int d\vec{r}_0\left(\frac{11}{3}\left[\ln\left(\frac{\vec{r}^{\,2}_{01}}{\vec{r}^{\,2}_{02}}\right)\left(\frac{1}{\vec{r}^{\,2}_{02}}-\frac{1}{\vec{r}^{\,2}_{01}}\right)-\frac{\vec{r}^{\,2}_{12}}{\vec{r}^{\,2}_{01}\vec{r}^{\,2}_{02}}\ln\left(\frac{\vec{r}^{\,2}_{12}}{\tilde{\mu}^2}\right)\right]\right.$$
$$\left.\times\left(B^{-}_{100}+B^{-}_{230}+B^{-}_{200}+B^{-}_{130}-B^{-}_{300}-B^{-}_{210}-B^{-}_{123}\right)+(1\leftrightarrow 3)+(2\leftrightarrow 3)\right). \tag{5.42}$$

6 Results

In this section we list the main results of the paper. In all the expressions $\overline{MS}$ renormalization scheme is used. Taking the LO equation from [10] (2.35) and using (3.30), we can write the NLO evolution equation for 3QWL operator as

$$\frac{\partial B_{123}}{\partial\eta}=\frac{\alpha_s(\mu^2)}{8\pi^2}\int d\vec{r}_0\left[(B_{100}B_{320}+B_{200}B_{310}-B_{300}B_{210}-6B_{123})\right.$$
$$\times\left\{\frac{\vec{r}^{\,2}_{12}}{\vec{r}^{\,2}_{01}\vec{r}^{\,2}_{02}}-\frac{3\alpha_s}{4\pi}\frac{11}{3}\left[\ln\left(\frac{\vec{r}^{\,2}_{01}}{\vec{r}^{\,2}_{02}}\right)\left(\frac{1}{\vec{r}^{\,2}_{02}}-\frac{1}{\vec{r}^{\,2}_{01}}\right)-\frac{\vec{r}^{\,2}_{12}}{\vec{r}^{\,2}_{01}\vec{r}^{\,2}_{02}}\ln\left(\frac{\vec{r}^{\,2}_{12}}{\tilde{\mu}^2}\right)\right]\right\}$$
$$-\frac{\alpha_s}{\pi}\ln\frac{\vec{r}^{\,2}_{20}}{\vec{r}^{\,2}_{21}}\ln\frac{\vec{r}^{\,2}_{10}}{\vec{r}^{\,2}_{21}}\left\{\frac{1}{2}\left[\frac{\vec{r}^{\,2}_{13}}{\vec{r}^{\,2}_{10}\vec{r}^{\,2}_{30}}-\frac{\vec{r}^{\,2}_{32}}{\vec{r}^{\,2}_{30}\vec{r}^{\,2}_{20}}\right](B_{100}B_{320}-B_{200}B_{310})\right.$$
$$\left.\left.-\frac{\vec{r}^{2}_{12}}{\vec{r}^{2}_{10}\vec{r}^{\,2}_{20}}\left(9B_{123}-\frac{1}{2}\left[2\left(B_{100}B_{320}+B_{200}B_{130}\right)-B_{300}B_{120}\right]\right)\right\}+(1\leftrightarrow 3)+(2\leftrightarrow 3)\right]$$

$$-\frac{\alpha_s^2}{8\pi^4}\int d\vec{r}_0 d\vec{r}_4 \ \Big[\{\tilde{L}_{12}\left(U_0U_4{}^{\dagger}U_2\right)\cdot\left(U_1U_0{}^{\dagger}U_4\right)\cdot U_3$$

$$+L_{12}\Big[\left(U_0U_4{}^{\dagger}U_2\right)\cdot\left(U_1U_0{}^{\dagger}U_4\right)\cdot U_3+tr\left(U_0U_4{}^{\dagger}\right)\left(U_1U_0{}^{\dagger}U_2\right)\cdot U_3\cdot U_4$$

$$-\frac{3}{4}[B_{144}B_{234}+B_{244}B_{134}-B_{344}B_{124}]+\frac{1}{2}B_{123}\Big]$$

$$+(M_{13}-M_{12}-M_{23}+M_2)\Big[\left(U_0U_4{}^{\dagger}U_3\right)\cdot\left(U_2U_0{}^{\dagger}U_1\right)\cdot U_4+\left(U_1U_0{}^{\dagger}U_2\right)\cdot\left(U_3U_4{}^{\dagger}U_0\right)\cdot U_4\Big]$$

$$+(\text{all 5 permutations } 1\leftrightarrow 2\leftrightarrow 3)\}+\ (0\leftrightarrow 4)\Big]\,. \tag{6.1}$$

Here the functions $L_{12}, \tilde{L}_{12}, M_{12}, M_2$ are defined in (3.9-3.12), the $\overline{MS}$ renormalization scale μ^2 is related to scale $\tilde{\mu}^2$ through (3.23).

The evolution equation for the composite 3QWL operator B_{123}^{conf} (4.2)

$$B_{123}^{conf}=B_{123}+\frac{\alpha_s 3}{8\pi^2}\int d\vec{r}_4\left[\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{41}^{\,2}\vec{r}_{42}^{\,2}}\ln\left(\frac{a\vec{r}_{12}^{\,2}}{\vec{r}_{41}^{\,2}\vec{r}_{42}^{\,2}}\right)\right.$$

$$\left.\times(-B_{123}+\frac{1}{6}(B_{144}B_{324}+B_{244}B_{314}-B_{344}B_{214}))+(1\leftrightarrow 3)+(2\leftrightarrow 3)\right] \tag{6.2}$$

follows from (4.20)

$$\frac{\partial B_{123}^{conf}}{\partial\eta}=\frac{\alpha_s\left(\mu^2\right)}{8\pi^2}\int d\vec{r}_0\ \Big[((B_{100}B_{320}+B_{200}B_{310}-B_{300}B_{210})-6B_{123})^{conf}$$

$$\times\left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}}-\frac{3\alpha_s}{4\pi}\frac{11}{3}\left[\ln\left(\frac{\vec{r}_{01}^{\,2}}{\vec{r}_{02}^{\,2}}\right)\left(\frac{1}{\vec{r}_{02}^{\,2}}-\frac{1}{\vec{r}_{01}^{\,2}}\right)-\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}}\ln\left(\frac{\vec{r}_{12}^{\,2}}{\tilde{\mu}^2}\right)\right]\right)+(1\leftrightarrow 3)+(2\leftrightarrow 3)\Big]$$

$$-\frac{\alpha_s^2}{32\pi^3}\int d\vec{r}_0\left(B_{003}B_{012}\left[\frac{\vec{r}_{32}{}^2}{\vec{r}_{03}{}^2\vec{r}_{02}{}^2}\ln^2\left(\frac{\vec{r}_{32}{}^2\vec{r}_{10}{}^2}{\vec{r}_{13}{}^2\vec{r}_{20}{}^2}\right)-\frac{\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{02}{}^2}\ln^2\left(\frac{\vec{r}_{12}{}^2\vec{r}_{30}{}^2}{\vec{r}_{13}{}^2\vec{r}_{20}{}^2}\right)\right]\right.$$

$$\left.+\,(\text{all 5 permutations } 1\leftrightarrow 2\leftrightarrow 3)\right)$$

$$-\frac{\alpha_s^2}{8\pi^4}\int d\vec{r}_0 d\vec{r}_4\ \left(\left\{\tilde{L}_{12}^C\left(U_0U_4{}^{\dagger}U_2\right)\cdot\left(U_1U_0{}^{\dagger}U_4\right)\cdot U_3\right.\right.$$

$$+L_{12}^C\Big[\left(U_0U_4{}^{\dagger}U_2\right)\cdot\left(U_1U_0{}^{\dagger}U_4\right)\cdot U_3+tr\left(U_0U_4{}^{\dagger}\right)\left(U_1U_0{}^{\dagger}U_2\right)\cdot U_3\cdot U_4$$

$$-\frac{3}{4}[B_{144}B_{234}+B_{244}B_{134}-B_{344}B_{124}]+\frac{1}{2}B_{123}\Big]$$

$$+M_{12}^C\Big[\left(U_0U_4{}^{\dagger}U_3\right)\cdot\left(U_2U_0{}^{\dagger}U_1\right)\cdot U_4+\left(U_1U_0{}^{\dagger}U_2\right)\cdot\left(U_3U_4{}^{\dagger}U_0\right)\cdot U_4\Big]$$

$$\left.\left.+\,(\text{all 5 permutations } 1\leftrightarrow 2\leftrightarrow 3)\right\}+(0\leftrightarrow 4)\right)\,. \tag{6.3}$$

Here the composite operator $((B_{100}B_{320}+B_{200}B_{310}-B_{300}B_{210})-6B_{123})^{conf}$ is defined in (4.5) according to model (4.1), the functions $L_{12}^C, \tilde{L}_{12}^C, M_{12}^C$ are defined in (4.15-4.17).

The equation for the composite 3QWL operator B_{123}^{conf} (4.2) linearized in the 3-gluon approximation is the result of (5.22) and (B.16)

$$\frac{\partial B_{123}^{conf}}{\partial\eta} \overset{3g}{=} \frac{3\alpha_s(\mu^2)}{4\pi^2}\int d\vec{r}_0\left[(B_{100}^{conf}+B_{320}^{conf}+B_{200}^{conf}+B_{310}^{conf}-B_{300}^{conf}-B_{210}^{conf}-B_{123}^{conf}-6)\right.$$
$$\left.\times\left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}}-\frac{3\alpha_s}{4\pi}\frac{11}{3}\left[\ln\left(\frac{\vec{r}_{01}^{\,2}}{\vec{r}_{02}^{\,2}}\right)\left(\frac{1}{\vec{r}_{02}^{\,2}}-\frac{1}{\vec{r}_{01}^{\,2}}\right)-\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}}\ln\left(\frac{\vec{r}_{12}^{\,2}}{\tilde{\mu}^2}\right)\right]\right)+(1\leftrightarrow3)+(2\leftrightarrow3)\right]$$
$$-\frac{9\alpha_s^2}{64\pi^3}\int d\vec{r}_0\left(\tilde{F}_{100}B_{100}+\tilde{F}_{230}B_{230}+(1\leftrightarrow3)+(1\leftrightarrow2)\right)$$
$$+\frac{27\alpha_s^2}{4\pi^2}\zeta(3)(3-\delta_{23}-\delta_{13}-\delta_{21})(B_{123}-6)-\frac{9\alpha_s^2}{8\pi^4}\int d\vec{r}_0 d\vec{r}_4(L_{12}^C+L_{13}^C+L_{23}^C)(B_{044}+B_{004}-12)$$
$$-\frac{9\alpha_s^2}{64\pi^4}\int d\vec{r}_0 d\vec{r}_4\ \left(\{F_0B_{040}+F_{140}B_{140}+(0\leftrightarrow4)\}+(\text{all 5 permutations}\,1\leftrightarrow2\leftrightarrow3)\right). \tag{6.4}$$

Here $\delta_{ij}=1$, if $\vec{r}_i=\vec{r}_j$ and $\delta_{ij}=0$ otherwise; the functions F_0 and F_{140} are defined in (5.18) and (5.19); $\tilde{F}_{100}$ and $\tilde{F}_{230}$ are defined in (5.24-5.25).

The linearized equations for C-even composite 3QWL Green function is the consequence of (5.33) and (B.16)

$$\frac{\partial B_{123}^{+conf}}{\partial\eta} \overset{3g}{=} -\frac{9\alpha_s^2}{64\pi^3}\int d\vec{r}_0\left(\tilde{F}_{100}B_{100}^{+}+\tilde{F}_{230}B_{230}^{+}+(1\leftrightarrow3)+(1\leftrightarrow2)\right)$$
$$+\frac{3\alpha_s(\mu^2)}{4\pi^2}\int d\vec{r}_0\left[(B_{100}^{+conf}+B_{320}^{+conf}+B_{200}^{+conf}+B_{310}^{+conf}-B_{300}^{+conf}-B_{210}^{+conf}-B_{123}^{+conf})\right.$$
$$\left.\times\left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}}-\frac{3\alpha_s}{4\pi}\frac{11}{3}\left[\ln\left(\frac{\vec{r}_{01}^{\,2}}{\vec{r}_{02}^{\,2}}\right)\left(\frac{1}{\vec{r}_{02}^{\,2}}-\frac{1}{\vec{r}_{01}^{\,2}}\right)-\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}}\ln\left(\frac{\vec{r}_{12}^{\,2}}{\tilde{\mu}^2}\right)\right]\right)+(1\leftrightarrow3)+(2\leftrightarrow3)\right]$$
$$+\frac{27\alpha_s^2}{4\pi^2}\zeta(3)(3-\delta_{23}-\delta_{13}-\delta_{21})B_{123}^{+}-\frac{9\alpha_s^2}{4\pi^4}\int d\vec{r}_0 d\vec{r}_4(L_{12}^C+L_{13}^C+L_{23}^C)B_{044}^{+}$$
$$-\frac{9\alpha_s^2}{64\pi^4}\int d\vec{r}_0 d\vec{r}_4\ \left(\{F_{140}+(0\leftrightarrow4)\}\,B_{140}^{+}+(\text{all 5 permutations}\,1\leftrightarrow2\leftrightarrow3)\right). \tag{6.5}$$

The linearized equations for C-odd composite 3QWL Green function is the consequence of (5.42) and (B.16)

$$\frac{\partial B_{123}^{-conf}}{\partial\eta} \overset{3g}{=} \frac{3\alpha_s(\mu^2)}{4\pi^2}\int d\vec{r}_0\left[\left(B_{100}^{-conf}+B_{320}^{-conf}+B_{200}^{-conf}+B_{310}^{-conf}\right.\right.$$
$$\left.-B_{300}^{-conf}-B_{210}^{-conf}-B_{123}^{-conf}\right)$$
$$\left.\times\left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}}-\frac{3\alpha_s}{4\pi}\frac{11}{3}\left[\ln\left(\frac{\vec{r}_{01}^{\,2}}{\vec{r}_{02}^{\,2}}\right)\left(\frac{1}{\vec{r}_{02}^{\,2}}-\frac{1}{\vec{r}_{01}^{\,2}}\right)-\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}}\ln\left(\frac{\vec{r}_{12}^{\,2}}{\tilde{\mu}^2}\right)\right]\right)+(1\leftrightarrow3)+(2\leftrightarrow3)\right]$$
$$-\frac{9\alpha_s^2}{64\pi^3}\int d\vec{r}_0\left(\tilde{F}_{100}B_{100}^{-}+\tilde{F}_{230}B_{230}^{-}+(1\leftrightarrow3)+(1\leftrightarrow2)\right)+\frac{27\alpha_s^2}{4\pi^2}\zeta(3)(3-\delta_{23}-\delta_{13}-\delta_{21})B_{123}^{-}$$

$$-\frac{9\alpha_s^2}{64\pi^4}\int d\vec{r}_0 d\vec{r}_4 \ \left(2F_0 B^-_{040} + \{F_{140} + (0 \leftrightarrow 4)\}\, B^-_{140} + (\text{all 5 permutations}\, 1 \leftrightarrow 2 \leftrightarrow 3)\right). \tag{6.6}$$

From these expressions one can see that terms with L_{ij}, L^C_{ij}, which comprise the BFKL kernels contribute only to the evolution of C-even part of the Green function while terms with $F_0, \tilde{L}_{ij}, \tilde{L}^C_{ij}$ contribute only to the evolution of the C-odd one.

7 Conclusion

In this paper we constructed the NLO evolution equation for the 3-quark Wilson loop operator $\varepsilon^{i'j'h'}\varepsilon_{ijh}U^i_{1i'}U^j_{2j'}U^h_{3h'}$. The kernel of this equation has nonconformal terms not related to renormalization. We found the composite 3QWL operator (4.2) obeying the NLO evolution equation with quasi-conformal kernel. We linearized the quasi-conformal equation in the 3-gluon approximation. Our result has correct dipole limit.

The 3QWL operator may have many phenomenological applications. First, it is a natural $SU(3)$ model for a baryon Green function in the Regge limit. Then, it is the irreducible operator describing C-odd exchange. For example as shown in the appendix D, the C-odd part of the quadrupole operator $tr(U_1U_2^\dagger U_3 U_4^\dagger)$ in the 3-gluon approximation in $SU(3)$ can be decomposed into a sum of 3QWLs

$$2tr(U_1U_2^\dagger U_3U_4^\dagger) - 2tr(U_4U_3^\dagger U_2U_1^\dagger) \overset{3g}{=} B^-_{144} + B^-_{322} - B^-_{433} - B^-_{211} + B^-_{124} + B^-_{234} - B^-_{123} - B^-_{134}.$$

The evolution equation for the C-odd part of the 3QWL operator is the generalization of the BKP equation for odderon exchange to the saturation regime. However, it is valid for the colorless object, i.e. for the function $B^-_{ijk} = B^-(\vec{r}_i, \vec{r}_j, \vec{r}_k)$, which vanishes as $\vec{r}_i = \vec{r}_j = \vec{r}_k$. The linear approximation of the equation for the C-odd part of the 3QWL should be equivalent to the NLO BKP for odderon exchange acting in the space of such functions. One may try to restore the full NLO BKP kernel from our result via the technique similar to the one developed for the 2-point operators in [16].

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A $SU(3)$ identities

Here we present the list of SU(3) identities used in the paper.

$$U_i \cdot U_j \cdot U_k = (U_iU_l^\dagger)\cdot(U_jU_l^\dagger)\cdot(U_kU_l^\dagger) = (U_l^\dagger U_i)\cdot(U_l^\dagger U_j)\cdot(U_l^\dagger U_k), \tag{A.1}$$

$$\varepsilon^{ijh}\varepsilon_{i'j'h'}(U_1)^{i'}_i(U_1)^{j'}_j = 2(U_1^\dagger)^h_{h'}, \quad U_1\cdot U_1\cdot U_3 = 2tr(U_1^\dagger U_3). \tag{A.2}$$

These identities follow from the definition of the group, namely from unitarity and the fact that the determinant of U is 1.

$$\left(U_2U_4^\dagger U_1 + U_1U_4^\dagger U_2\right)\cdot U_4\cdot U_3 = -B_{123} + \frac{1}{2}(B_{144}B_{324} + B_{244}B_{314} - B_{344}B_{214}). \tag{A.3}$$

This identity can be checked using (A.1) with $l = 4$ and then expanding the product of Levi-Civita symbols as

$$\varepsilon_{ijh}\varepsilon^{i'j'h'} = \begin{vmatrix} \delta_i^{i'} & \delta_i^{j'} & \delta_i^{h'} \\ \delta_j^{i'} & \delta_j^{j'} & \delta_j^{h'} \\ \delta_h^{i'} & \delta_h^{j'} & \delta_h^{h'} \end{vmatrix}. \tag{A.4}$$

$$\begin{aligned} 0 = [&\left(U_0U_4{}^\dagger U_3U_0{}^\dagger U_4\right)\cdot U_1\cdot U_2 - U_1\cdot U_2\cdot U_4 tr\left(U_0{}^\dagger U_3\right)tr\left(U_4{}^\dagger U_0\right) \\ &+tr\left(U_0U_4{}^\dagger\right)\left(U_2U_0{}^\dagger U_3 + U_3U_0{}^\dagger U_2\right)\cdot U_1\cdot U_4 \\ &+\left(U_0U_4{}^\dagger U_2\right)\cdot\left(U_3U_0{}^\dagger U_4\right)\cdot U_1 + \left(U_0U_4{}^\dagger U_3\right)\cdot\left(U_2U_0{}^\dagger U_4\right)\cdot U_1 + (1\leftrightarrow 2)] + (4\leftrightarrow 0). \end{aligned} \tag{A.5}$$

This identity relates the color structures in $\mathbf{G}_{12\langle 3\rangle}$, $\mathbf{G}_{1\langle 23\rangle}$ and $\mathbf{G}_{\langle 13\rangle 2}$. By $1 \leftrightarrow 2 \leftrightarrow 3$ transformation one can obtain 2 more identities and totally eliminate 3 color structures from $\mathbf{G}_{12\langle 3\rangle}$, $\mathbf{G}_{\langle 1\rangle 23}$, and $\mathbf{G}_{1\langle 2\rangle 3}$.

$$\begin{aligned} 0 = &tr\left(U_0{}^\dagger U_2\right)\left(U_0U_4{}^\dagger U_3 + U_3U_4{}^\dagger U_0\right)\cdot U_1\cdot U_4 \\ &-\left(U_0U_4{}^\dagger U_3\right)\cdot\left(U_2U_0{}^\dagger U_4\right)\cdot U_1 - \left(U_3U_4{}^\dagger U_0\right)\cdot\left(U_4U_0{}^\dagger U_2\right)\cdot U_1 \\ &-tr\left(U_4U_0{}^\dagger\right)\left(U_2U_4{}^\dagger U_3 + U_3U_4{}^\dagger U_2\right)\cdot U_0\cdot U_1 \\ &+\left(U_3U_4{}^\dagger U_2U_0{}^\dagger U_4\right)\cdot U_0\cdot U_1 + \left(U_4U_0{}^\dagger U_2U_4{}^\dagger U_3\right)\cdot U_0\cdot U_1 \\ &-\left(U_0U_4{}^\dagger U_3\right)\cdot\left(U_2U_0{}^\dagger U_1\right)\cdot U_4 - \left(U_1U_0{}^\dagger U_2\right)\cdot\left(U_3U_4{}^\dagger U_0\right)\cdot U_4 \\ &+\left(U_1U_0{}^\dagger U_4\right)\cdot\left(U_3U_4{}^\dagger U_2\right)\cdot U_0 + \left(U_2U_4{}^\dagger U_3\right)\cdot\left(U_4U_0{}^\dagger U_1\right)\cdot U_0. \end{aligned} \tag{A.6}$$

This identity relates all color structures in $\mathbf{G}_{1\langle 23\rangle}$ and two structures in $\mathbf{G}_{\langle 123\rangle}$. It goes into 5 different identities after $1 \leftrightarrow 2 \leftrightarrow 3$ transformation, which allows one to get rid of 6 structures.

$$\begin{aligned} 0 = &tr\left(U_0{}^\dagger U_2\right)\left(U_0U_4{}^\dagger U_1 + U_1U_4{}^\dagger U_0\right)\cdot U_3\cdot U_4 \\ &+\left(U_0U_4{}^\dagger U_2U_0{}^\dagger U_1 + U_1U_0{}^\dagger U_2U_4{}^\dagger U_0\right)\cdot U_3\cdot U_4 \\ &-tr\left(U_0U_4{}^\dagger\right)\left(U_1U_0{}^\dagger U_2 + U_2U_0{}^\dagger U_1\right)\cdot U_3\cdot U_4 \\ &-\left(U_0U_4{}^\dagger U_1\right)\cdot\left(U_2U_0{}^\dagger U_4\right)\cdot U_3 - \left(U_0U_4{}^\dagger U_2\right)\cdot\left(U_1U_0{}^\dagger U_4\right)\cdot U_3 \\ &-\left(U_1U_0{}^\dagger U_4\right)\cdot\left(U_3U_4{}^\dagger U_2\right)\cdot U_0 + \left(U_1U_4{}^\dagger U_2\right)\cdot\left(U_3U_0{}^\dagger U_4\right)\cdot U_0 \\ &+\left(U_2U_4{}^\dagger U_1\right)\cdot\left(U_4U_0{}^\dagger U_3\right)\cdot U_0 - \left(U_2U_4{}^\dagger U_3\right)\cdot\left(U_4U_0{}^\dagger U_1\right)\cdot U_0 + (4\leftrightarrow 0). \end{aligned} \tag{A.7}$$

This identity relates 2 color structures in $\mathbf{G}_{\langle 12\rangle 3}$ and a structure in $\mathbf{G}_{\langle 123\rangle}$. It also goes into 5 different identities after $1 \leftrightarrow 2 \leftrightarrow 3$ transformation, which allows one to get rid of 6 structures.

$$0 = [U_0\cdot U_1\cdot U_2 tr\left(U_0{}^\dagger U_4\right)tr\left(U_4{}^\dagger U_3\right)$$

$$-tr\left(U_4{U_0}^\dagger\right)\left(U_1{U_4}^\dagger U_3+U_3{U_4}^\dagger U_1\right)\cdot U_0\cdot U_2+\left(U_0{U_4}^\dagger U_1\right)\cdot\left(U_3{U_0}^\dagger U_4\right)\cdot U_2$$

$$+\left(U_1{U_4}^\dagger U_0\right)\cdot\left(U_4{U_0}^\dagger U_3\right)\cdot U_2+(1\leftrightarrow 2)]-(4\leftrightarrow 0). \qquad (A.8)$$

This identity relates 2 color structures in $\mathbf{G}_{\langle 13\rangle 2}$, 2 color structures in $\mathbf{G}_{1\langle 23\rangle}$ and a structure in $\mathbf{G}_{12\langle 3\rangle}$. It goes into 2 different identities after $1\leftrightarrow 2\leftrightarrow 3$ transformation, which allows one to get rid of 3 more structures.

$$0=2tr\left(U_4{U_0}^\dagger\right)\left(U_2{U_4}^\dagger U_3+U_3{U_4}^\dagger U_2\right)\cdot U_0\cdot U_1$$

$$+\left(U_0{U_4}^\dagger U_1+U_1{U_4}^\dagger U_0\right)\cdot\left(U_2{U_0}^\dagger U_3+U_3{U_0}^\dagger U_2\right)\cdot U_4$$

$$+\left(U_0{U_4}^\dagger U_2-U_2{U_4}^\dagger U_0\right)\cdot\left(U_3{U_0}^\dagger U_1-U_1{U_0}^\dagger U_3\right)\cdot U_4$$

$$+\left(U_0{U_4}^\dagger U_3-U_3{U_4}^\dagger U_0\right)\cdot\left(U_2{U_0}^\dagger U_1-U_1{U_0}^\dagger U_2\right)\cdot U_4-(4\leftrightarrow 0). \qquad (A.9)$$

This identity relates 3 color structures in $\mathbf{G}_{\langle 132\rangle}$ and a color structure in $\mathbf{G}_{1\langle 23\rangle}$. It goes into 2 different identities after $1\leftrightarrow 2\leftrightarrow 3$ transformation, which allows one to get rid of 3 more structures.

All these identities (A.5–A.9) can be checked using (A.1) with $l=1$ and then expanding the product of Levi-Civita symbols via (A.4).

$$0=2tr({U_0}^\dagger U_3)\left(U_1{U_4}^\dagger U_2\right)\cdot U_0\cdot U_4-\left(U_1{U_4}^\dagger U_2\right)\cdot\left(U_3{U_0}^\dagger U_4\right)\cdot U_0$$

$$-\left(U_1{U_4}^\dagger U_2\right)\cdot\left(U_4{U_0}^\dagger U_3\right)\cdot U_0-2\left(U_1{U_4}^\dagger U_2\right)\cdot U_3\cdot U_4$$

$$-\left(U_1{U_4}^\dagger U_2{U_0}^\dagger U_3\right)\cdot U_0\cdot U_4-\left(U_3{U_0}^\dagger U_1{U_4}^\dagger U_2\right)\cdot U_0\cdot U_4+(1\leftrightarrow 2). \qquad (A.10)$$

This identity can be proved directly using (A.3).

$$0=2tr\left({U_0}^\dagger U_4\right)\left(U_1{U_4}^\dagger U_2\right)\cdot U_0\cdot U_3-tr\left({U_0}^\dagger U_1\right)\left(U_0{U_4}^\dagger U_2+U_2{U_4}^\dagger U_0\right)\cdot U_3\cdot U_4$$

$$+\left(U_0{U_4}^\dagger U_1\right)\cdot\left(U_2{U_0}^\dagger U_3\right)\cdot U_4+\left(U_0{U_4}^\dagger U_1\right)\cdot\left(U_2{U_0}^\dagger U_4\right)\cdot U_3$$

$$+\left(U_1{U_4}^\dagger U_0\right)\cdot\left(U_3{U_0}^\dagger U_2\right)\cdot U_4+\left(U_1{U_4}^\dagger U_0\right)\cdot\left(U_4{U_0}^\dagger U_2\right)\cdot U_3$$

$$-\left(U_1{U_4}^\dagger U_2\right)\cdot\left(U_3{U_0}^\dagger U_4+U_4{U_0}^\dagger U_3\right)\cdot U_0$$

$$-\left(U_1{U_4}^\dagger U_2{U_0}^\dagger U_4\right)\cdot U_0\cdot U_3-\left(U_4{U_0}^\dagger U_1{U_4}^\dagger U_2\right)\cdot U_0\cdot U_3+(1\leftrightarrow 2). \qquad (A.11)$$

$$0=tr\left({U_0}^\dagger U_2\right)\left(U_0{U_4}^\dagger U_1+U_1{U_4}^\dagger U_0\right)\cdot U_3\cdot U_4-2\left(U_1{U_4}^\dagger U_2+U_2{U_4}^\dagger U_1\right)\cdot U_3\cdot U_4$$

$$+U_0\cdot U_3\cdot U_4tr\left({U_0}^\dagger U_1{U_4}^\dagger U_2\right)+U_0\cdot U_3\cdot U_4tr\left({U_0}^\dagger U_2{U_4}^\dagger U_1\right)$$

$$-\left(U_1{U_4}^\dagger U_2{U_0}^\dagger U_3\right)\cdot U_0\cdot U_4-\left(U_3{U_0}^\dagger U_2{U_4}^\dagger U_1\right)\cdot U_0\cdot U_4$$

$$-\left(U_1{U_4}^\dagger U_2{U_0}^\dagger U_4\right)\cdot U_0\cdot U_3-\left(U_4{U_0}^\dagger U_2{U_4}^\dagger U_1\right)\cdot U_0\cdot U_3$$

$$
\begin{aligned}
&-\left(U_0{U_4}^\dagger U_1\right)\cdot\left(U_2{U_0}^\dagger U_3\right)\cdot U_4-\left(U_1{U_4}^\dagger U_0\right)\cdot\left(U_3{U_0}^\dagger U_2\right)\cdot U_4\\
&-\left(U_0{U_4}^\dagger U_1\right)\cdot\left(U_2{U_0}^\dagger U_4\right)\cdot U_3-\left(U_1{U_4}^\dagger U_0\right)\cdot\left(U_4{U_0}^\dagger U_2\right)\cdot U_3.
\end{aligned}
\tag{A.12}
$$

$$
\begin{aligned}
0=tr\left({U_4}^\dagger U_1\right)&\left(U_2{U_0}^\dagger U_4+U_4{U_0}^\dagger U_2\right)\cdot U_0\cdot U_3\\
+\left(U_0{U_4}^\dagger U_1\right)&\cdot\left(U_3{U_0}^\dagger U_2\right)\cdot U_4+\left(U_1{U_4}^\dagger U_0\right)\cdot\left(U_2{U_0}^\dagger U_3\right)\cdot U_4\\
+\left(U_2{U_0}^\dagger U_1{U_4}^\dagger U_0\right)&\cdot U_3\cdot U_4-\left(U_4{U_0}^\dagger U_2{U_4}^\dagger U_1\right)\cdot U_0\cdot U_3-(1\leftrightarrow 2,0\leftrightarrow 4).
\end{aligned}
\tag{A.13}
$$

These identities (A.11–A.13) also can be checked using (A.1) with $l=3$ and then expanding the product of Levi-Civita symbols via (A.4).

$$
\begin{aligned}
0=&\left(U_2{U_4}^\dagger U_0-U_0{U_4}^\dagger U_2\right)\cdot\left(U_3{U_0}^\dagger U_1-U_1{U_0}^\dagger U_3\right)\cdot U_4\\
&+\left(U_3{U_4}^\dagger U_2-U_2{U_4}^\dagger U_3\right)\cdot\left(U_4{U_0}^\dagger U_1-U_1{U_0}^\dagger U_4\right)\cdot U_0\\
&+\left(U_1{U_0}^\dagger U_2-U_2{U_0}^\dagger U_1\right)\cdot\left(U_3{U_4}^\dagger U_0-U_0{U_4}^\dagger U_3\right)\cdot U_4+(0\leftrightarrow 4).
\end{aligned}
\tag{A.14}
$$

It can be proved using (A.1) with $l=4$ and (A.4).

B Construction of conformal 4-point operator

Here we derive the evolution equation for the operator

$$
\left(\left(U_2U_4^\dagger U_1+U_1U_4^\dagger U_2\right)\cdot U_4\cdot U_3-2B_{123}\right)=(-3B_{123}+\frac{1}{2}(B_{144}B_{324}+B_{244}B_{314}-B_{344}B_{214})).
\tag{B.1}
$$

So one has to find the evolution of the operator $\left(U_1U_4^\dagger U_2\right)\cdot U_4\cdot U_3$ first. It reads

$$
\begin{aligned}
\frac{\partial\left(U_1U_4^\dagger U_2\right)\cdot U_4\cdot U_3}{\partial\eta}&=\left(U_1U_4^\dagger U_2\right)\cdot U_4\cdot U_3\left(-\frac{\alpha_s}{2\pi^2}\frac{4}{3}2\right)\int d\vec{z}_0\left(\frac{1}{\vec{r}_{10}^{\,2}}+\frac{1}{\vec{r}_{20}^{\,2}}+\frac{1}{\vec{r}_{30}^{\,2}}+\frac{2}{\vec{r}_{40}^{\,2}}\right)\\
&-\frac{\alpha_s}{\pi^2}\left[\left(t^cU_1U_4^\dagger t^cU_2\right)\cdot U_4\cdot U_3+\left(U_1t^cU_4^\dagger U_2t^c\right)\cdot U_4\cdot U_3\right]\int d\vec{r}_0\frac{(\vec{r}_{10}\vec{r}_{20})}{\vec{r}_{10}^{\,2}\vec{r}_{20}^{\,2}}\\
&-\frac{\alpha_s}{\pi^2}\left[\left(t^cU_1U_4^\dagger U_2\right)\cdot U_4\cdot(t^cU_3)+\left(U_1t^cU_4^\dagger U_2\right)\cdot U_4\cdot(U_3t^c)\right]\int d\vec{r}_0\frac{(\vec{r}_{10}\vec{r}_{30})}{\vec{r}_{10}^{\,2}\vec{r}_{30}^{\,2}}\\
&-\frac{\alpha_s}{\pi^2}\left[\left(U_1U_4^\dagger t^cU_2\right)\cdot U_4\cdot(t^cU_3)+\left(U_1U_4^\dagger U_2t^c\right)\cdot U_4\cdot(U_3t^c)\right]\int d\vec{r}_0\frac{(\vec{r}_{20}\vec{r}_{30})}{\vec{r}_{20}^{\,2}\vec{r}_{30}^{\,2}}\\
&-\frac{\alpha_s}{\pi^2}\left[\left(t^cU_1U_4^\dagger U_2\right)\cdot(t^cU_4)+\left(U_1t^cU_4^\dagger U_2\right)\cdot(U_4t^c)\right.\\
&\left.-\left(t^cU_1U_4^\dagger t^cU_2\right)\cdot U_4-\left(U_1t^ct^cU_4^\dagger U_2\right)\cdot U_4\right]\cdot U_3\int d\vec{r}_0\frac{(\vec{r}_{10}\vec{r}_{40})}{\vec{r}_{10}^{\,2}\vec{r}_{40}^{\,2}}\\
&-\frac{\alpha_s}{\pi^2}\left[\left(U_1U_4^\dagger t^cU_2\right)\cdot(t^cU_4)+\left(U_1U_4^\dagger U_2t^c\right)\cdot(U_4t^c)\right]
\end{aligned}
$$

$$-\Big(U_1U_4^\dagger t^c t^c U_2\Big)\cdot U_4-\Big(U_1t^cU_4^\dagger U_2 t^c\Big)\cdot U_4\Big]\cdot U_3\int d\vec r_0\frac{(\vec r_{20}\vec r_{40})}{\vec r_{20}^{\,2}\vec r_{40}^{\,2}}$$

$$-\frac{\alpha_s}{\pi^2}\Big[\Big(U_1U_4^\dagger U_2\Big)\cdot(t^cU_4)\cdot(t^cU_3)+\Big(U_1U_4^\dagger U_2\Big)\cdot(U_4t^c)\cdot(U_3t^c)$$

$$-\Big(U_1U_4^\dagger t^cU_2\Big)\cdot U_4\cdot(t^cU_3)-\Big(U_1t^cU_4^\dagger U_2\Big)\cdot U_4\cdot(U_3t^c)\Big]\int d\vec r_0\frac{(\vec r_{30}\vec z_{40})}{\vec r_{30}^{\,2}\vec r_{40}^{\,2}}$$

$$+\frac{\alpha_s}{\pi^2}\Big[\Big(U_1U_4^\dagger t^cU_2\Big)\cdot(t^cU_4)\cdot U_3+\Big(U_1t^cU_4^\dagger U_2\Big)\cdot(U_4t^c)\cdot U_3\Big]\int\frac{d\vec r_0}{\vec r_{40}^{\,2}}$$

$$+\frac{\alpha_s}{\pi^2}\int d\vec r_0U_0^{cd}\left(\frac{\Big(U_1U_4^\dagger U_2\Big)\cdot(t^cU_4t^d)\cdot U_3}{\vec r_{04}^{\,2}}+\frac{\Big(U_1t^dU_4^\dagger t^cU_2\Big)\cdot U_4\cdot U_3}{\vec r_{04}^{\,2}}\right.$$

$$\left.+\frac{\Big(t^cU_1t^dU_4^\dagger U_2\Big)\cdot U_4\cdot U_3}{\vec r_{01}^{\,2}}+\frac{\Big(U_1U_4^\dagger t^cU_2t^d\Big)\cdot U_4\cdot U_3}{\vec r_{02}^{\,2}}+\frac{\Big(U_1U_4^\dagger U_2\Big)\cdot U_4\cdot(t^cU_3t^d)}{\vec r_{03}^{\,2}}\right)$$

$$+\frac{\alpha_s}{\pi^2}\int d\vec r_0\frac{(\vec r_{03}\vec z_{02})}{\vec r_{03}^{\,2}\vec r_{02}^{\,2}}U_0^{cd}\Big(\Big(U_1U_4^\dagger t^cU_2\Big)\cdot U_4\cdot(U_3t^d)+\Big(U_1U_4^\dagger U_2t^d\Big)\cdot U_4\cdot(t^cU_3)\Big)$$

$$+\frac{\alpha_s}{\pi^2}\int d\vec r_0\frac{(\vec r_{01}\vec z_{02})}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}U_0^{cd}\Big(\Big(U_1t^dU_4^\dagger t^cU_2\Big)\cdot U_4\cdot U_3+\Big(t^cU_1U_4^\dagger U_2t^d\Big)\cdot U_4\cdot U_3\Big)$$

$$+\frac{\alpha_s}{\pi^2}\int d\vec r_0\frac{(\vec r_{01}\vec z_{03})}{\vec r_{01}^{\,2}\vec r_{03}^{\,2}}U_0^{cd}\Big(\Big(U_1t^dU_4^\dagger U_2\Big)\cdot U_4\cdot(t^cU_3)+\Big(t^cU_1U_4^\dagger U_2\Big)\cdot U_4\cdot(U_3t^d)\Big)$$

$$+\frac{\alpha_s}{\pi^2}\int d\vec r_0\frac{(\vec r_{04}\vec z_{01})}{\vec r_{04}^{\,2}\vec r_{01}^{\,2}}U_0^{cd}\Big(\Big(U_1t^dU_4^\dagger U_2\Big)\cdot(t^cU_4)\cdot U_3+\Big(t^cU_1U_4^\dagger U_2\Big)\cdot(U_4t^d)\cdot U_3$$

$$-\Big(U_1t^dU_4^\dagger t^cU_2\Big)\cdot U_4\cdot U_3-\Big(t^cU_1t^dU_4^\dagger U_2\Big)\cdot U_4\cdot U_3\Big)$$

$$+\frac{\alpha_s}{\pi^2}\int d\vec r_0\frac{(\vec r_{04}\vec z_{02})}{\vec r_{04}^{\,2}\vec r_{02}^{\,2}}U_0^{cd}\Big(\Big(U_1U_4^\dagger U_2t^d\Big)\cdot(t^cU_4)\cdot U_3+\Big(U_1U_4^\dagger t^cU_2\Big)\cdot(U_4t^d)\cdot U_3$$

$$-\Big(U_1U_4^\dagger t^cU_2t^d\Big)\cdot U_4\cdot U_3-\Big(U_1t^dU_4^\dagger t^cU_2\Big)\cdot U_4\cdot U_3\Big)$$

$$+\frac{\alpha_s}{\pi^2}\int d\vec r_0\frac{(\vec r_{04}\vec z_{03})}{\vec r_{04}^{\,2}\vec r_{03}^{\,2}}U_0^{cd}\Big(\Big(U_1U_4^\dagger U_2\Big)\cdot(t^cU_4)\cdot(U_3t^d)+\Big(U_1U_4^\dagger U_2\Big)\cdot(U_4t^d)\cdot(t^cU_3)$$

$$-\Big(U_1U_4^\dagger t^cU_2\Big)\cdot U_4\cdot(U_3t^d)-\Big(U_1t^dU_4^\dagger U_2\Big)\cdot U_4\cdot(t^cU_3)\Big)$$

$$-\frac{\alpha_s}{\pi^2}\int\frac{d\vec r_0}{\vec r_{04}^{\,2}}U_0^{cd}\Big(\Big(U_1t^dU_4^\dagger U_2\Big)\cdot(t^cU_4)\cdot U_3+\Big(U_1U_4^\dagger t^cU_2\Big)\cdot(U_4t^d)\cdot U_3\Big).\qquad\text{(B.2)}$$

Using (A.3) and (A.10–A.12) after the convolution one gets

$$\frac{\partial}{\partial\eta}\Big(\Big(U_2U_4^\dagger U_1+U_1U_4^\dagger U_2\Big)\cdot U_4\cdot U_3-2B_{123}\Big)=\frac{\alpha_s}{4\pi^2}\int d\vec r_0$$

$$\times\left(A_{34}\frac{\vec r_{34}^{\,2}}{\vec r_{03}^{\,2}\vec r_{04}^{\,2}}+A_{13}\frac{\vec r_{13}^{\,2}}{\vec r_{03}^{\,2}\vec r_{01}^{\,2}}+A_{23}\frac{\vec r_{23}^{\,2}}{\vec r_{03}^{\,2}\vec r_{02}^{\,2}}+A_{14}\frac{\vec r_{14}^{\,2}}{\vec r_{01}^{\,2}\vec r_{04}^{\,2}}+A_{24}\frac{\vec r_{24}^{\,2}}{\vec r_{02}^{\,2}\vec r_{04}^{\,2}}+A_{12}\frac{\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}\right).\qquad\text{(B.3)}$$

Here

$$A_{34} = -2\left(U_2 U_4^\dagger U_1 + U_1 U_4^\dagger U_2\right)\cdot U_4\cdot U_3 + \left(U_3 U_4^\dagger U_1 + U_1 U_4^\dagger U_3\right)\cdot U_4\cdot U_2$$
$$+\left(U_3 U_4^\dagger U_2 + U_2 U_4^\dagger U_3\right)\cdot U_4\cdot U_1 + \left(U_2 {U_4}^\dagger U_1 + U_1 {U_4}^\dagger U_2\right)\cdot\left(U_3 {U_0}^\dagger U_4 + U_4 {U_0}^\dagger U_3\right)\cdot U_0$$
$$-\left(U_2 {U_4}^\dagger U_0\right)\cdot\left(U_3 {U_0}^\dagger U_1\right)\cdot U_4 - \left(U_0 {U_4}^\dagger U_2\right)\cdot\left(U_1 {U_0}^\dagger U_3\right)\cdot U_4$$
$$-\left(U_0 {U_4}^\dagger U_1\right)\cdot\left(U_2 {U_0}^\dagger U_3\right)\cdot U_4 - \left(U_1 {U_4}^\dagger U_0\right)\cdot\left(U_3 {U_0}^\dagger U_2\right)\cdot U_4. \quad \text{(B.4)}$$

$$A_{13} = \left(U_0 {U_4}^\dagger U_2\right)\cdot\left(U_1 {U_0}^\dagger U_3\right)\cdot U_4 + \left(U_2 {U_4}^\dagger U_0\right)\cdot\left(U_3 {U_0}^\dagger U_1\right)\cdot U_4$$
$$+\left(U_3 {U_0}^\dagger U_1 {U_4}^\dagger U_2\right)\cdot U_0\cdot U_4 + \left(U_2 {U_4}^\dagger U_1 {U_0}^\dagger U_3\right)\cdot U_0\cdot U_4$$
$$-2\left(U_1 {U_0}^\dagger U_3 + U_3 {U_0}^\dagger U_1\right)\cdot U_0\cdot U_2 - \left(U_3 {U_4}^\dagger U_2 + U_2 {U_4}^\dagger U_3\right)\cdot U_1\cdot U_4$$
$$-\left(U_1 {U_4}^\dagger U_2 + U_2 {U_4}^\dagger U_1\right)\cdot U_3\cdot U_4 + 4U_1\cdot U_2\cdot U_3. \quad \text{(B.5)}$$

$$A_{14} = tr\left({U_0}^\dagger U_1\right)\left(U_2 {U_4}^\dagger U_0 + U_0 {U_4}^\dagger U_2\right)\cdot U_3\cdot U_4 + tr\left({U_4}^\dagger U_0\right)\left(U_1 {U_0}^\dagger U_2 + U_2 {U_0}^\dagger U_1\right)\cdot U_3\cdot U_4$$
$$+\left(U_2 {U_4}^\dagger U_0\right)\cdot\left(U_4 {U_0}^\dagger U_1\right)\cdot U_3 + \left(U_0 {U_4}^\dagger U_2\right)\cdot\left(U_1 {U_0}^\dagger U_4\right)\cdot U_3$$
$$-2U_4\cdot U_2\cdot U_3 tr\left({U_4}^\dagger U_1\right) - 2U_1\cdot U_2\cdot U_3 - 4\left(U_1 {U_4}^\dagger U_2 + U_2 {U_4}^\dagger U_1\right)\cdot U_3\cdot U_4$$
$$+\left(U_4 {U_0}^\dagger U_1 {U_4}^\dagger U_2\right)\cdot U_0\cdot U_3 + \left(U_2 {U_4}^\dagger U_1 {U_0}^\dagger U_4\right)\cdot U_0\cdot U_3. \quad \text{(B.6)}$$

$$A_{12} = -2\left(U_1 {U_0}^\dagger U_2 + U_2 {U_0}^\dagger U_1\right)\cdot U_3\cdot U_0 - tr\left({U_4}^\dagger U_0\right)\left(U_1 {U_0}^\dagger U_2 + U_2 {U_0}^\dagger U_1\right)\cdot U_3\cdot U_4$$
$$+4U_1\cdot U_2\cdot U_3 + 2U_4\cdot U_2\cdot U_3 tr\left({U_4}^\dagger U_1\right) + 2U_4\cdot U_1\cdot U_3 tr\left({U_4}^\dagger U_2\right)$$
$$-U_0\cdot U_3\cdot U_4\left(tr\left({U_0}^\dagger U_1 {U_4}^\dagger U_2\right) + tr\left({U_0}^\dagger U_2 {U_4}^\dagger U_1\right)\right). \quad \text{(B.7)}$$

$$A_{23} = A_{13}|_{\vec r_1\leftrightarrow\vec r_2}, \quad A_{24} = A_{14}|_{\vec r_1\leftrightarrow\vec r_2}. \quad \text{(B.8)}$$

Our model for the composite conformal operators reads

$$O^{conf} = O + \frac{1}{2}\frac{\partial O}{\partial\eta}\Bigg|_{\frac{\vec r_{mn}^{\,2}}{\vec r_{im}^{\,2}\vec r_{in}^{\,2}}\to\frac{\vec r_{mn}^{\,2}}{\vec r_{im}^{\,2}\vec r_{in}^{\,2}}\ln\left(\frac{\vec r_{mn}^{\,2}a}{\vec r_{im}^{\,2}\vec r_{in}^{\,2}}\right)}, \quad \text{(B.9)}$$

where a is an arbitrary constant. Thus

$$B_{123}^{conf} = B_{123} + \frac{1}{2}\frac{\partial B_{123}}{\partial\eta}\Bigg|_{\frac{\vec r_{mn}^{\,2}}{\vec r_{im}^{\,2}\vec r_{in}^{\,2}}\to\frac{\vec r_{mn}^{\,2}}{\vec r_{im}^{\,2}\vec r_{in}^{\,2}}\ln\left(\frac{\vec r_{mn}^{\,2}a}{\vec r_{im}^{\,2}\vec r_{in}^{\,2}}\right)} \quad \text{(B.10)}$$
$$= B_{123} + \frac{\alpha_s 3}{8\pi^2}\int d\vec r_0\left[\frac{\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}\ln\left(\frac{a\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}\right)\right.$$
$$\left.\times(-B_{123} + \frac{1}{6}(B_{100}B_{320} + B_{200}B_{310} - B_{300}B_{210})) + (1\leftrightarrow 3) + (2\leftrightarrow 3)\right], \quad \text{(B.11)}$$

$$(-3B_{123}+\frac{1}{2}(B_{144}B_{324}+B_{244}B_{314}-B_{344}B_{214}))^{conf}$$
$$=(-3B_{123}+\frac{1}{2}(B_{144}B_{324}+B_{244}B_{314}-B_{344}B_{214}))$$
$$+\frac{1}{2}\frac{\partial}{\partial\eta}(-3B_{123}+\frac{1}{2}(B_{144}B_{324}+B_{244}B_{314}-B_{344}B_{214}))\Bigg|_{\frac{\vec r_{mn}^2}{\vec r_{im}^2\vec r_{in}^2}\to\frac{\vec r_{mn}^2}{\vec r_{im}^2\vec r_{in}^2}\ln\left(\frac{\vec r_{mn}^2 a}{\vec r_{im}^2\vec r_{in}^2}\right)} \tag{B.12}$$
$$=(-3B_{123}+\frac{1}{2}(B_{144}B_{324}+B_{244}B_{314}-B_{344}B_{214}))$$
$$+\frac{\alpha_s}{8\pi^2}\int d\vec r_0\left(A_{34}\frac{\vec r_{34}^{\,2}}{\vec r_{03}^{\,2}\vec r_{04}^{\,2}}\ln\left(\frac{\vec r_{34}^{\,2}a}{\vec r_{03}^{\,2}\vec r_{04}^{\,2}}\right)+A_{13}\frac{\vec r_{13}^{\,2}}{\vec r_{03}^{\,2}\vec r_{01}^{\,2}}\ln\left(\frac{\vec r_{13}^{\,2}a}{\vec r_{03}^{\,2}\vec r_{01}^{\,2}}\right)+A_{23}\frac{\vec r_{23}^{\,2}}{\vec r_{03}^{\,2}\vec r_{02}^{\,2}}\ln\left(\frac{\vec r_{23}^{\,2}a}{\vec r_{03}^{\,2}\vec r_{02}^{\,2}}\right)\right.$$
$$\left.+A_{14}\frac{\vec r_{14}^{\,2}}{\vec r_{01}^{\,2}\vec r_{04}^{\,2}}\ln\left(\frac{\vec r_{14}^{\,2}a}{\vec r_{01}^{\,2}\vec r_{04}^{\,2}}\right)+A_{24}\frac{\vec r_{24}^{\,2}}{\vec r_{02}^{\,2}\vec z_{04}^{\,2}}\ln\left(\frac{\vec r_{24}^{\,2}a}{\vec r_{02}^{\,2}\vec z_{04}^{\,2}}\right)+A_{12}\frac{\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}\ln\left(\frac{\vec r_{12}^{\,2}a}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}\right)\right). \tag{B.13}$$

In the 3-gluon approximation

$$(-3B_{123}+\frac{1}{2}(B_{144}B_{324}+B_{244}B_{314}-B_{344}B_{214}))$$
$$\stackrel{3g}{=}3(-B_{123}+B_{144}+B_{324}+B_{244}+B_{314}-B_{344}-B_{214}-6). \tag{B.14}$$

Therefore

$$(-3B_{123}+\frac{1}{2}(B_{144}B_{324}+B_{244}B_{314}-B_{344}B_{214}))^{conf}$$
$$\stackrel{3g}{=}3(-B_{123}+B_{144}+B_{324}+B_{244}+B_{314}-B_{344}-B_{214}-6)$$
$$+\frac{3}{2}\frac{\partial}{\partial\eta}(-B_{123}+B_{144}+B_{324}+B_{244}+B_{314}-B_{344}-B_{214}-6)\Bigg|_{\frac{\vec r_{mn}^2}{\vec r_{im}^2\vec r_{in}^2}\to\frac{\vec r_{mn}^2}{\vec r_{im}^2\vec r_{in}^2}\ln\left(\frac{\vec r_{mn}^2 a}{\vec r_{im}^2\vec r_{in}^2}\right)} \tag{B.15}$$
$$=3(-B_{123}^{conf}+B_{144}^{conf}+B_{324}^{conf}+B_{244}^{conf}+B_{314}^{conf}-B_{344}^{conf}-B_{214}^{conf}-6). \tag{B.16}$$

C Integrals

Here we describe the calculation of integral (4.19). It reads

$$\int d\vec r_4 Z_{12}=J_{12}-(1\leftrightarrow 3). \tag{C.1}$$

$$J_{12}=\frac{\vec r_{12}{}^2}{8\vec r_{01}{}^2\vec r_{02}{}^2}\int d\vec r_4\left[\left(\frac{\vec r_{03}^{\,2}}{\vec r_{04}^{\,2}\vec r_{34}^{\,2}}-\frac{\vec r_{02}^{\,2}}{\vec r_{04}^{\,2}\vec r_{24}^{\,2}}\right)\ln\left(\frac{\vec r_{02}{}^2\vec r_{14}{}^2}{\vec r_{04}{}^2\vec r_{12}{}^2}\right)\right.$$
$$\left.+\frac{\vec r_{01}{}^2}{\vec r_{04}{}^2\vec r_{14}{}^2}\ln\left(\frac{\vec r_{02}{}^2\vec r_{34}{}^2}{\vec r_{03}{}^2\vec r_{24}{}^2}\right)+\frac{\vec r_{13}{}^2}{\vec r_{14}{}^2\vec r_{34}{}^2}\ln\left(\frac{\vec r_{03}{}^2\vec r_{12}{}^2}{\vec r_{02}{}^2\vec r_{13}{}^2}\right)\right]. \tag{C.2}$$

Since the integral is conformally invariant, one can set $\vec r_0=0$ and make the inversion, then calculate the integral and then again make the inversion and restore $\vec r_0$.

$$J_{12}\stackrel{\vec r_0=0}{\to}\frac{\vec r_{12}{}^2}{8\vec r_1{}^2\vec r_2{}^2}\int dr_4\left[\left(\frac{\vec r_3{}^2}{\vec r_4{}^2\vec r_{34}{}^2}-\frac{\vec r_2{}^2}{\vec r_4{}^2\vec r_{24}{}^2}\right)\ln\left(\frac{\vec r_2{}^2\vec r_{14}{}^2}{\vec r_4{}^2\vec r_{12}{}^2}\right)\right.$$

$$+\frac{\vec{r}_1{}^2}{\vec{r}_4{}^2\vec{r}_{14}{}^2}\ln\left(\frac{\vec{r}_2{}^2\vec{r}_{34}{}^2}{\vec{r}_3{}^2\vec{r}_{24}{}^2}\right)+\frac{\vec{r}_{13}{}^2}{\vec{r}_{14}{}^2\vec{r}_{34}{}^2}\ln\left(\frac{\vec{r}_3{}^2\vec{r}_{12}{}^2}{\vec{r}_2{}^2\vec{r}_{13}{}^2}\right)\Bigg] \tag{C.3}$$

$$\overset{\text{inversion}}{\to}\frac{r_{12}^2}{8}\int dr_4\left[\left(\frac{1}{\vec{r}_{34}{}^2}-\frac{1}{\vec{r}_{24}{}^2}\right)\ln\left(\frac{\vec{r}_{14}{}^2}{\vec{r}_{12}{}^2}\right)+\frac{1}{\vec{r}_{14}{}^2}\ln\left(\frac{\vec{r}_{34}{}^2}{\vec{r}_{24}{}^2}\right)+\frac{\vec{r}_{13}{}^2}{\vec{r}_{14}{}^2\vec{r}_{34}{}^2}\ln\left(\frac{\vec{r}_{12}{}^2}{\vec{r}_{13}{}^2}\right)\right]. \tag{C.4}$$

Using the integrals from appendix A in [17] we have

$$J_{12}\to-\pi\frac{r_{12}^2}{8}\ln^2\left(\frac{\vec{r}_{13}^{\,2}}{\vec{r}_{12}^{\,2}}\right). \tag{C.5}$$

After inversion and restoring of $\vec{r}_0$ we get

$$J_{12}=\ -\pi\frac{\vec{r}_{12}{}^2}{8\vec{r}_{01}{}^2\vec{r}_{02}{}^2}\ln^2\left(\frac{\vec{r}_{12}{}^2\vec{r}_{30}{}^2}{\vec{r}_{13}{}^2\vec{r}_{20}{}^2}\right). \tag{C.6}$$

Therefore

$$\int\frac{d\vec{r}_4}{\pi}Z_{12}=\frac{\vec{r}_{32}{}^2}{8\vec{r}_{03}{}^2\vec{r}_{02}{}^2}\ln^2\left(\frac{\vec{r}_{32}{}^2\vec{r}_{10}{}^2}{\vec{r}_{13}{}^2\vec{r}_{20}{}^2}\right)-\frac{\vec{r}_{12}{}^2}{8\vec{r}_{01}{}^2\vec{r}_{02}{}^2}\ln^2\left(\frac{\vec{r}_{12}{}^2\vec{r}_{30}{}^2}{\vec{r}_{13}{}^2\vec{r}_{20}{}^2}\right). \tag{C.7}$$

Now we will integrate F_{100} (5.20) w.r.t. $\vec{r}_4$. Again we set $\vec{r}_0=0$, do inversion, and calculate the integral in the $d=2+2\epsilon$ dimensional space using the integrals from appendix A in [17] and

$$\int\frac{d^{2+2\epsilon}r_{14}}{\pi^{1+\epsilon}\Gamma(1-\epsilon)}\frac{r_{34}{}^2}{r_{14}{}^2r_{24}{}^2}=\frac{r_{13}^2+r_{23}^2-r_{12}^2}{r_{12}^2}\left(\frac{1}{\epsilon}+\ln\left(r_{12}^2\right)\right)+O(\epsilon). \tag{C.8}$$

We get

$$\int\frac{d\vec{r}_4}{\pi}F_{100}+(2\leftrightarrow3)\to\int\frac{d^dr_4}{\pi}\left(\frac{r_{12}{}^2}{r_{24}{}^2}\ln\left(\frac{r_{14}{}^2}{r_{12}{}^2}\right)+\frac{r_{23}{}^2r_{12}{}^2}{2r_{14}{}^2r_{24}{}^2}\ln\left(\frac{r_{14}{}^2r_{23}{}^2}{r_{12}{}^2r_{24}{}^2}\right)\right.$$
$$+\frac{r_{13}{}^2r_{12}{}^2}{r_{14}{}^2r_{24}{}^2}\ln\left(\frac{r_{13}{}^2r_{24}{}^2}{r_{12}{}^2r_{14}{}^2}\right)-\frac{r_{13}{}^2}{r_{24}{}^2}\ln\left(\frac{r_{13}{}^2r_{24}{}^2}{r_{14}{}^4}\right)+\frac{r_{23}{}^2}{2r_{24}{}^2}\ln\left(\frac{r_{23}{}^2r_{24}{}^4r_{34}{}^2}{r_{14}{}^8}\right)$$
$$\left.+\frac{r_{23}{}^2}{2r_{14}{}^2}\ln\left(\frac{r_{24}{}^2r_{34}{}^2}{r_{14}{}^2r_{23}{}^2}\right)-\frac{r_{34}{}^2r_{12}{}^2}{2r_{14}{}^2r_{24}{}^2}\ln\left(\frac{r_{24}{}^2r_{34}{}^6}{r_{12}{}^6r_{14}{}^2}\right)-\frac{r_{12}{}^2}{2r_{14}{}^2}\ln\left(\frac{r_{34}{}^4}{r_{12}{}^2r_{14}{}^2}\right)\right)+(2\leftrightarrow3) \tag{C.9}$$

$$\overset{d\to2}{\to}\left(\frac{3\left(r_{13}{}^2-r_{12}{}^2\right)}{4}-\frac{r_{23}{}^2}{2}\right)\ln^2\left(\frac{r_{12}{}^2}{r_{23}{}^2}\right)+\left(\frac{3\left(r_{12}{}^2-r_{13}{}^2\right)}{4}-\frac{r_{23}{}^2}{2}\right)\ln^2\left(\frac{r_{13}{}^2}{r_{23}{}^2}\right)$$
$$+\left(\frac{3}{4}r_{23}{}^2-r_{12}{}^2-r_{13}{}^2\right)\ln^2\left(\frac{r_{12}{}^2}{r_{13}{}^2}\right)+\frac{3}{2}S_{123}I\left(r_{12}{}^2,r_{13}{}^2,r_{23}{}^2\right). \tag{C.10}$$

Here

$$S_{123}=r_{12}{}^4+r_{13}{}^4+r_{23}{}^4-2r_{13}{}^2r_{12}{}^2-2r_{23}{}^2r_{12}{}^2-2r_{13}{}^2r_{23}{}^2 \tag{C.11}$$

is the Cayley-Menger determinant proportional to the squared area of the triangle with the corners at $r=r_{1,2,3}$, and

$$I(a,b,c)=\int_0^1\frac{dx}{a(1-x)+bx-cx(1-x)}\ln\left(\frac{a(1-x)+bx}{cx(1-x)}\right) \tag{C.12}$$

$$= \int_0^1 \int_0^1 \int_0^1 \frac{dx_1 dx_2 dx_3 \delta(1-x_1-x_2-x_3)}{(ax_1+bx_2+cx_3)(x_1x_2+x_1x_3+x_2x_3)} \tag{C.13}$$

$$= \int_0^1 dx \int_0^1 dz \, \frac{1}{cx(1-x)z+(b(1-x)+ax)(1-z)} \, . \tag{C.14}$$

is symmetric w.r.t. interchange of its arguments function defined in [18]. Performing inversion and restoring r_0, we get

$$\int \frac{d\vec{r}_4}{\pi} F_{100} + (2 \leftrightarrow 3) = \left(\frac{3\vec{r}_{23}^{\,2}}{4\vec{r}_{02}^{\,2}\vec{r}_{03}^{\,2}} - \frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}} - \frac{\vec{r}_{13}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{03}^{\,2}} \right) \ln^2 \left(\frac{\vec{r}_{03}^{\,2}\vec{r}_{12}^{\,2}}{\vec{r}_{02}^{\,2}\vec{r}_{13}^{\,2}} \right)$$
$$+ \left(\frac{3\vec{r}_{12}^{\,2}}{4\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}} - \frac{3\vec{r}_{13}^{\,2}}{4\vec{r}_{01}^{\,2}\vec{r}_{03}^{\,2}} - \frac{\vec{r}_{23}^{\,2}}{2\vec{r}_{02}^{\,2}\vec{r}_{03}^{\,2}} \right) \ln^2 \left(\frac{\vec{r}_{02}^{\,2}\vec{r}_{13}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{23}^{\,2}} \right)$$
$$+ \left(\frac{3\vec{r}_{13}^{\,2}}{4\vec{r}_{01}^{\,2}\vec{r}_{03}^{\,2}} - \frac{3\vec{r}_{12}^{\,2}}{4\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}} - \frac{\vec{r}_{23}^{\,2}}{2\vec{r}_{02}^{\,2}\vec{r}_{03}^{\,2}} \right) \ln^2 \left(\frac{\vec{r}_{03}^{\,2}\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{23}^{\,2}} \right)$$
$$+ \frac{3}{2} \tilde{S}_{123} I \left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}}, \frac{\vec{r}_{13}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{03}^{\,2}}, \frac{\vec{r}_{23}^{\,2}}{\vec{r}_{02}^{\,2}\vec{r}_{03}^{\,2}} \right)$$
$$+ X \left(\frac{\vec{r}_{02}^{\,2}\vec{r}_{13}^{\,2}}{\vec{r}_{03}^{\,2}\vec{r}_{12}^{\,2}}, \frac{\vec{r}_{02}^{\,2}\vec{r}_{13}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{23}^{\,2}} \right) \delta(\vec{r}_{20}) + X \left(\frac{\vec{r}_{03}^{\,2}\vec{r}_{12}^{\,2}}{\vec{r}_{02}^{\,2}\vec{r}_{13}^{\,2}}, \frac{\vec{r}_{03}^{\,2}\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{23}^{\,2}} \right) \delta(\vec{r}_{30})$$
$$+ Y \left(\frac{\vec{r}_{01}^{\,2}\vec{r}_{23}^{\,2}}{\vec{r}_{03}^{\,2}\vec{r}_{12}^{\,2}}, \frac{\vec{r}_{01}^{\,2}\vec{r}_{23}^{\,2}}{\vec{r}_{02}^{\,2}\vec{r}_{13}^{\,2}} \right) \delta\left(\vec{r}_{10}\right). \tag{C.15}$$

Here

$$\tilde{S}_{123} = \left(\frac{\vec{r}_{12}^{\,4}}{\vec{r}_{01}^{\,4}\vec{r}_{02}^{\,4}} + \frac{\vec{r}_{13}^{\,4}}{\vec{r}_{01}^{\,4}\vec{r}_{03}^{\,4}} + \frac{\vec{r}_{23}^{\,4}}{\vec{r}_{02}^{\,4}\vec{r}_{03}^{\,4}} - \frac{2\vec{r}_{13}^{\,2}\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,4}\vec{r}_{02}^{\,2}\vec{r}_{03}^{\,2}} - \frac{2\vec{r}_{23}^{\,2}\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,4}\vec{r}_{03}^{\,2}} - \frac{2\vec{r}_{13}^{\,2}\vec{r}_{23}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}\vec{r}_{03}^{\,4}} \right) \tag{C.16}$$

and we added the delta-functional contributions, which may be lost via inversion. Thanks to conformal invariance of the integral such contributions may depend only on conformally invariant ratios. We can find the values of the unknown functions X and Y at $\vec{r}_2 = \vec{r}_3, \vec{r}_2 = \vec{r}_1, \vec{r}_1 = \vec{r}_3$. Using (5.9) we have

$$\int \frac{d\vec{r}_4}{\pi} F_{100} + (2 \leftrightarrow 3)|_{\vec{r}_2=\vec{r}_3} = 16 \int \frac{d\vec{r}_4}{\pi} \tilde{L}_{12}^C = 24\pi\zeta(3)[\delta(\vec{r}_{10}) - \delta(\vec{r}_{20})]$$
$$= 2X\left(1,\infty\right)\delta(\vec{r}_{20}) + Y\left(0,0\right)\delta\left(\vec{r}_{10}\right). \tag{C.17}$$

Therefore

$$X\left(1,\infty\right) = -12\pi\zeta\left(3\right), \quad Y\left(0,0\right) = 24\pi\zeta(3). \tag{C.18}$$

$$\int \frac{d\vec{r}_4}{\pi} F_{100} + (2 \leftrightarrow 3)|_{\vec{r}_1=\vec{r}_3} = -4 \int \frac{d\vec{r}_4}{\pi} \tilde{L}_{12}^C = -6\pi\zeta(3)[\delta(\vec{r}_{10}) - \delta(\vec{r}_{20})]$$
$$= X\left(0,0\right)\delta(\vec{r}_{20}) + (Y\left(1,\infty\right) + X\left(\infty,1\right))\delta\left(\vec{r}_{10}\right). \tag{C.19}$$

Here again we used (5.9). Therefore

$$X\left(0,0\right) = 6\pi\zeta\left(3\right), \quad Y\left(1,\infty\right) + X\left(\infty,1\right) = -6\pi\zeta(3). \tag{C.20}$$

If $\vec r_2 \neq \vec r_3, \vec r_2 \neq \vec r_1, \vec r_1 \neq \vec r_3$ then the arguments of X and Y are fixed by the integration w.r.t. $\vec r_0$

$$X\left(\frac{\vec r_{02}{}^2\vec r_{13}{}^2}{\vec r_{03}{}^2\vec r_{12}{}^2}, \frac{\vec r_{02}{}^2\vec r_{13}{}^2}{\vec r_{01}{}^2\vec r_{23}{}^2}\right)\delta(\vec r_{20}) = X\left(0,0\right)\delta(\vec r_{20}) = 6\pi\zeta\left(3\right)\delta(\vec r_{20}), \tag{C.21}$$

$$Y\left(\frac{\vec r_{01}{}^2\vec r_{23}{}^2}{\vec r_{03}{}^2\vec r_{12}{}^2}, \frac{\vec r_{01}{}^2\vec r_{23}{}^2}{\vec r_{02}{}^2\vec r_{13}{}^2}\right)\delta\left(\vec r_{10}\right) = Y\left(0,0\right)\delta\left(\vec r_{10}\right) = 24\pi\zeta(3)\delta\left(\vec r_{10}\right). \tag{C.22}$$

As a result, one can write

$$\begin{aligned}
\int \frac{d\vec r_4}{\pi} F_{100} + (2\leftrightarrow 3) = &\left(\frac{3\vec r_{23}{}^2}{4\vec r_{02}{}^2\vec r_{03}{}^2} - \frac{\vec r_{12}{}^2}{\vec r_{01}{}^2\vec r_{02}{}^2} - \frac{\vec r_{13}{}^2}{\vec r_{01}{}^2\vec r_{03}{}^2}\right)\ln^2\left(\frac{\vec r_{03}{}^2\vec r_{12}{}^2}{\vec r_{02}{}^2\vec r_{13}{}^2}\right)\\
&+\left(\frac{3\vec r_{12}{}^2}{4\vec r_{01}{}^2\vec r_{02}{}^2} - \frac{3\vec r_{13}{}^2}{4\vec r_{01}{}^2\vec r_{03}{}^2} - \frac{\vec r_{23}{}^2}{2\vec r_{02}{}^2\vec r_{03}{}^2}\right)\ln^2\left(\frac{\vec r_{02}{}^2\vec r_{13}{}^2}{\vec r_{01}{}^2\vec r_{23}{}^2}\right)\\
&+\left(\frac{3\vec r_{13}{}^2}{4\vec r_{01}{}^2\vec r_{03}{}^2} - \frac{3\vec r_{12}{}^2}{4\vec r_{01}{}^2\vec r_{02}{}^2} - \frac{\vec r_{23}{}^2}{2\vec r_{02}{}^2\vec r_{03}{}^2}\right)\ln^2\left(\frac{\vec r_{03}{}^2\vec r_{12}{}^2}{\vec r_{01}{}^2\vec r_{23}{}^2}\right)\\
&+\frac{3}{2}\tilde S_{123} I\left(\frac{\vec r_{12}{}^2}{\vec r_{01}{}^2\vec r_{02}{}^2}, \frac{\vec r_{13}{}^2}{\vec r_{01}{}^2\vec r_{03}{}^2}, \frac{\vec r_{23}{}^2}{\vec r_{02}{}^2\vec r_{03}{}^2}\right)\\
&+6\pi\zeta\left(3\right)\left(\delta(\vec r_{20}) + \delta(\vec r_{30})\right) + 24\pi\zeta(3)\delta\left(\vec r_{10}\right)\\
&- 36\pi\zeta\left(3\right)\delta_{23}\delta(\vec r_{20}) - 36\pi\zeta(3)(\delta_{13}+\delta_{12})\delta\left(\vec r_{10}\right) + 72\pi\zeta(3)\delta_{13}\delta_{12}\delta\left(\vec r_{10}\right).
\end{aligned} \tag{C.23}$$

Here $\delta_{ij} = 1$, if $\vec r_i = \vec r_j$ and $\delta_{ij} = 0$ otherwise. The last term is added since the total contribution at $\vec r_1 = \vec r_2 = \vec r_3$ is 0.

Now we will integrate F_{230} (5.21) w.r.t. $\vec r_4$. Again we set $\vec r_0 = 0$, do inversion, and calculate the integral in the d-dimensional space using the integrals from appendix A in [17] and (C.8). We get

$$\begin{aligned}
\int \frac{d\vec r_4}{\pi} F_{230} + (2\leftrightarrow 3) \to \int \frac{d^d r_4}{\pi} &\left(\frac{r_{34}{}^2 r_{12}{}^2}{2r_{14}{}^2 r_{24}{}^2}\ln\left(\frac{r_{14}{}^2 r_{34}{}^6}{r_{24}{}^2 r_{12}{}^6}\right) + \frac{r_{12}{}^2}{2r_{14}{}^2}\ln\left(\frac{r_{12}{}^2 r_{34}{}^8}{r_{14}{}^4 r_{24}{}^6}\right)\right.\\
&-\frac{r_{12}{}^2}{2r_{24}{}^2}\ln\left(\frac{r_{14}{}^2}{r_{12}{}^2}\right) - \frac{r_{23}{}^2 r_{12}{}^2}{r_{14}{}^2 r_{24}{}^2}\ln\left(\frac{r_{14}{}^2 r_{23}{}^2}{r_{12}{}^2 r_{24}{}^2}\right) - \frac{r_{13}{}^2 r_{12}{}^2}{2r_{14}{}^2 r_{24}{}^2}\ln\left(\frac{r_{13}{}^2 r_{24}{}^2}{r_{12}{}^2 r_{14}{}^2}\right)\\
&\left.-\frac{r_{23}{}^2}{r_{14}{}^2}\ln\left(\frac{r_{24}{}^2 r_{34}{}^2}{r_{14}{}^2 r_{23}{}^2}\right) + \frac{r_{13}{}^2}{2r_{24}{}^2}\ln\left(\frac{r_{13}{}^2 r_{24}{}^2}{r_{14}{}^4}\right) - \frac{r_{23}{}^2}{2r_{24}{}^2}\ln\left(\frac{r_{23}{}^4 r_{24}{}^2}{r_{14}{}^4 r_{34}{}^2}\right)\right) + (2\leftrightarrow 3)
\end{aligned} \tag{C.24}$$

$$\begin{aligned}
\overset{d\to 2}{\to} &\left(\frac{3\left(r_{12}{}^2 - r_{13}{}^2\right)}{4} + r_{23}{}^2\right)\ln^2\left(\frac{r_{12}{}^2}{r_{23}{}^2}\right) + \left(\frac{3\left(r_{13}{}^2 - r_{12}{}^2\right)}{4} + r_{23}{}^2\right)\ln^2\left(\frac{r_{13}{}^2}{r_{23}{}^2}\right)\\
&+\left(\frac{r_{12}{}^2 + r_{13}{}^2}{2} - \frac{3}{4}r_{23}{}^2\right)\ln^2\left(\frac{r_{12}{}^2}{r_{13}{}^2}\right) - \frac{3}{2}S_{123} I\left(r_{12}{}^2, r_{13}{}^2, r_{23}{}^2\right).
\end{aligned} \tag{C.25}$$

Again, inverting and restoring r_0 we have

$$\begin{aligned}
\int \frac{d\vec r_4}{\pi} F_{230} + (2\leftrightarrow 3) = &\left(\frac{\vec r_{12}{}^2}{2\vec r_{01}{}^2\vec r_{02}{}^2} + \frac{\vec r_{13}{}^2}{2\vec r_{01}{}^2\vec r_{03}{}^2} - \frac{3\vec r_{23}{}^2}{4\vec r_{02}{}^2\vec r_{03}{}^2}\right)\ln^2\left(\frac{\vec r_{03}{}^2\vec r_{12}{}^2}{\vec r_{02}{}^2\vec r_{13}{}^2}\right)\\
&+\left(\frac{3\vec r_{13}{}^2}{4\vec r_{01}{}^2\vec r_{03}{}^2} - \frac{3\vec r_{12}{}^2}{4\vec r_{01}{}^2\vec r_{02}{}^2} + \frac{\vec r_{23}{}^2}{\vec r_{02}{}^2\vec r_{03}{}^2}\right)\ln^2\left(\frac{\vec r_{02}{}^2\vec r_{13}{}^2}{\vec r_{01}{}^2\vec r_{23}{}^2}\right)
\end{aligned}$$

$$+\left(\frac{3\vec{r}_{12}{}^2}{4\vec{r}_{01}{}^2\vec{r}_{02}{}^2}-\frac{3\vec{r}_{13}{}^2}{4\vec{r}_{01}{}^2\vec{r}_{03}{}^2}+\frac{\vec{r}_{23}{}^2}{\vec{r}_{02}{}^2\vec{r}_{03}{}^2}\right)\ln^2\left(\frac{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}\right)$$
$$-\frac{3}{2}\tilde{S}_{123}I\left(\frac{\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{02}{}^2},\frac{\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{03}{}^2},\frac{\vec{r}_{23}{}^2}{\vec{r}_{02}{}^2\vec{r}_{03}{}^2}\right)$$
$$+\tilde{X}\left(\frac{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}{\vec{r}_{03}{}^2\vec{r}_{12}{}^2},\frac{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}\right)\delta(\vec{r}_{20})+\tilde{X}\left(\frac{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}{\vec{r}_{02}{}^2\vec{r}_{13}{}^2},\frac{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}\right)\delta(\vec{r}_{30})$$
$$+\tilde{Y}\left(\frac{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}{\vec{r}_{03}{}^2\vec{r}_{12}{}^2},\frac{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}\right)\delta\left(\vec{r}_{10}\right). \tag{C.26}$$

Again, we can find the values of $\tilde{X}$ and $\tilde{Y}$ putting $\vec{r}_2=\vec{r}_3, \vec{r}_2=\vec{r}_1, \vec{r}_1=\vec{r}_3$ in this equation. Indeed via (5.9) we have,

$$\int\frac{d\vec{r}_4}{\pi}F_{230}+(2\leftrightarrow 3)|_{\vec{r}_2=\vec{r}_3}=-8\int\frac{d\vec{r}_4}{\pi}\tilde{L}^C_{12}=-12\pi\zeta(3)[\delta(\vec{r}_{10})-\delta(\vec{r}_{20})]$$
$$=2\tilde{X}\left(1,\infty\right)\delta(\vec{r}_{20})+\tilde{Y}\left(0,0\right)\delta\left(\vec{r}_{10}\right). \tag{C.27}$$

Therefore

$$\tilde{X}\left(1,\infty\right)=6\pi\zeta\left(3\right),\quad \tilde{Y}\left(0,0\right)=-12\pi\zeta(3). \tag{C.28}$$

Using (5.9) again, we get

$$\int\frac{d\vec{r}_4}{\pi}F_{230}+(2\leftrightarrow 3)|_{\vec{r}_1=\vec{r}_3}=8\int\frac{d\vec{r}_4}{\pi}\tilde{L}^C_{12}=12\pi\zeta(3)[\delta(\vec{r}_{10})-\delta(\vec{r}_{20})]$$
$$=\tilde{X}\left(0,0\right)\delta(\vec{r}_{20})+\Big(\tilde{X}\left(\infty,1\right)+\tilde{Y}\left(1,\infty\right)\Big)\delta\left(\vec{r}_{10}\right). \tag{C.29}$$

Therefore

$$\tilde{X}\left(0,0\right)=-12\pi\zeta\left(3\right),\quad \tilde{Y}\left(1,\infty\right)+\tilde{X}\left(\infty,1\right)=12\pi\zeta(3). \tag{C.30}$$

If $\vec{r}_2\neq\vec{r}_3, \vec{r}_2\neq\vec{r}_1, \vec{r}_1\neq\vec{r}_3$ then the arguments of $\tilde{X}$ and $\tilde{Y}$ are fixed by the integration w.r.t. $\vec{r}_0$

$$\tilde{X}\left(\frac{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}{\vec{r}_{03}{}^2\vec{r}_{12}{}^2},\frac{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}\right)\delta(\vec{r}_{20})=\tilde{X}\left(0,0\right)\delta(\vec{r}_{20})=-12\pi\zeta\left(3\right)\delta(\vec{r}_{20}), \tag{C.31}$$

$$\tilde{Y}\left(\frac{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}{\vec{r}_{03}{}^2\vec{r}_{12}{}^2},\frac{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}\right)\delta\left(\vec{r}_{10}\right)=\tilde{Y}\left(0,0\right)\delta\left(\vec{r}_{10}\right)=-12\pi\zeta(3)\delta\left(\vec{r}_{10}\right). \tag{C.32}$$

Finally,

$$\int\frac{d\vec{r}_4}{\pi}F_{230}+(2\leftrightarrow 3)=\left(\frac{\vec{r}_{12}{}^2}{2\vec{r}_{01}{}^2\vec{r}_{02}{}^2}+\frac{\vec{r}_{13}{}^2}{2\vec{r}_{01}{}^2\vec{r}_{03}{}^2}-\frac{3\vec{r}_{23}{}^2}{4\vec{r}_{02}{}^2\vec{r}_{03}{}^2}\right)\ln^2\left(\frac{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}\right)$$
$$+\left(\frac{3\vec{r}_{13}{}^2}{4\vec{r}_{01}{}^2\vec{r}_{03}{}^2}-\frac{3\vec{r}_{12}{}^2}{4\vec{r}_{01}{}^2\vec{r}_{02}{}^2}+\frac{\vec{r}_{23}{}^2}{\vec{r}_{02}{}^2\vec{r}_{03}{}^2}\right)\ln^2\left(\frac{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}\right)$$
$$+\left(\frac{3\vec{r}_{12}{}^2}{4\vec{r}_{01}{}^2\vec{r}_{02}{}^2}-\frac{3\vec{r}_{13}{}^2}{4\vec{r}_{01}{}^2\vec{r}_{03}{}^2}+\frac{\vec{r}_{23}{}^2}{\vec{r}_{02}{}^2\vec{r}_{03}{}^2}\right)\ln^2\left(\frac{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}\right)$$
$$-\frac{3}{2}\tilde{S}_{123}I\left(\frac{\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{02}{}^2},\frac{\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{03}{}^2},\frac{\vec{r}_{23}{}^2}{\vec{r}_{02}{}^2\vec{r}_{03}{}^2}\right)$$

$$-12\pi\zeta\,(3)\,(\delta(\vec{r}_{20})+\delta(\vec{r}_{30})+\delta\,(\vec{r}_{10}))$$
$$+\,36\pi\zeta\,(3)\,\delta_{23}\delta(\vec{r}_{20})+36\pi\zeta(3)(\delta_{13}+\delta_{12})\delta\,(\vec{r}_{10})-72\pi\zeta(3)\delta_{13}\delta_{12}\delta\,(\vec{r}_{10})\,. \tag{C.33}$$

Now we will integrate (5.19) and prove equality (5.41). Again we set $\vec{r}_0=0$, do inversion, and calculate the integral in the d-dimensional space using the integrals from appendix A in [17] and (C.8). We get

$$\int\frac{d\vec{r}_4}{\pi}\ \left(\{F_{140}+(0\leftrightarrow 4)\}+(2\leftrightarrow 3)\right)\to\int\frac{d^dr_4}{\pi}\left(\frac{{r_{12}}^2}{{r_{14}}^2}\ln\left(\frac{{r_{12}}^2{r_{34}}^4}{{r_{14}}^2{r_{24}}^4}\right)\right.$$
$$+\frac{{r_{12}}^2}{{r_{24}}^2}\ln\left(\frac{{r_{12}}^2{r_{24}}^2}{{r_{34}}^4}\right)-\frac{{r_{23}}^2{r_{12}}^2}{{r_{14}}^2{r_{24}}^2}\ln\left(\frac{{r_{14}}^2{r_{23}}^2}{{r_{12}}^2{r_{24}}^2}\right)-\frac{{r_{23}}^2{r_{12}}^2}{{r_{24}}^2{r_{34}}^2}\ln\left(\frac{{r_{23}}^2{r_{24}}^2}{{r_{12}}^2{r_{34}}^2}\right)$$
$$-\frac{{r_{23}}^2}{{r_{14}}^2}\ln\left(\frac{{r_{24}}^2{r_{34}}^2}{{r_{14}}^2{r_{23}}^2}\right)+\frac{{r_{13}}^2}{{r_{24}}^2}\ln\left(\frac{{r_{34}}^4}{{r_{13}}^2{r_{24}}^2}\right)+\frac{{r_{23}}^2}{{r_{24}}^2}\ln\left(\frac{{r_{34}}^2}{{r_{23}}^2}\right)$$
$$\left.+\frac{{r_{23}}^2}{{r_{34}}^2}\ln\left(\frac{{r_{24}}^2}{{r_{23}}^2}\right)+\frac{{r_{13}}^2{r_{24}}^2}{{r_{14}}^2{r_{34}}^2}\ln\left(\frac{{r_{14}}^2{r_{24}}^2}{{r_{13}}^2{r_{34}}^2}\right)-\frac{{r_{14}}^2{r_{23}}^2}{{r_{24}}^2{r_{34}}^2}\ln\left(\frac{{r_{14}}^2}{{r_{23}}^2}\right)\right)+(2\leftrightarrow 3) \tag{C.34}$$
$$\overset{d\to 2}{\to}-\frac{{r_{13}}^2+{r_{12}}^2}{2}\ln^2\left(\frac{{r_{12}}^2}{{r_{13}}^2}\right)+\frac{1}{2}{r_{23}}^2\left(\ln^2\left(\frac{{r_{12}}^2}{{r_{23}}^2}\right)+\ln^2\left(\frac{{r_{13}}^2}{{r_{23}}^2}\right)\right). \tag{C.35}$$

Inverting and restoring $\vec{r}_0$, we get

$$\int\frac{d\vec{r}_4}{\pi}\ \left(\{F_{140}+(0\leftrightarrow 4)\}+(2\leftrightarrow 3)\right)=-\frac{1}{2}\left(\frac{{\vec{r}_{12}}^{\,2}}{{\vec{r}_{01}}^{\,2}{\vec{r}_{02}}^{\,2}}+\frac{{\vec{r}_{13}}^{\,2}}{{\vec{r}_{01}}^{\,2}{\vec{r}_{03}}^{\,2}}\right)\ln^2\left(\frac{{\vec{r}_{03}}^{\,2}{\vec{r}_{12}}^{\,2}}{{\vec{r}_{02}}^{\,2}{\vec{r}_{13}}^{\,2}}\right)$$
$$+\frac{{\vec{r}_{23}}^{\,2}}{2{\vec{r}_{02}}^{\,2}{\vec{r}_{03}}^{\,2}}\ln^2\left(\frac{{\vec{r}_{03}}^{\,2}{\vec{r}_{12}}^{\,2}}{{\vec{r}_{01}}^{\,2}{\vec{r}_{23}}^{\,2}}\right)+\frac{{\vec{r}_{23}}^{\,2}}{2{\vec{r}_{02}}^{\,2}{\vec{r}_{03}}^{\,2}}\ln^2\left(\frac{{\vec{r}_{02}}^{\,2}{\vec{r}_{13}}^{\,2}}{{\vec{r}_{01}}^{\,2}{\vec{r}_{23}}^{\,2}}\right). \tag{C.36}$$

This integral has no delta functional contributions since it equals 0 at $\vec{r}_1=\vec{r}_2, \vec{r}_1=\vec{r}_3, \vec{r}_3=\vec{r}_2$.

D Decomposition of C-odd quadrupole operator

Here we demonstrate that the C-odd part of the quadrupole operator $tr(U_1U_2^\dagger U_3U_4^\dagger)$ in the 3-gluon approximation in $SU(3)$ can be decomposed into a sum of 3QWLs. Indeed

$$2tr(U_1U_2^\dagger U_3U_4^\dagger)=\Big((U_1-U_2)(U_2^\dagger-U_3^\dagger)(U_3-U_4)\Big)\cdot U_4\cdot U_4$$
$$-B_{133}+B_{233}+B_{144}-B_{244}+B_{344}+B_{122}-6 \tag{D.1}$$
$$\overset{3g}{=}-(U_1-U_2)(U_2-U_3)(U_3-U_4)\cdot E\cdot E-B_{133}+B_{233}+B_{144}-B_{244}+B_{344}+B_{122}-6. \tag{D.2}$$

Therefore

$$2tr(U_1U_2^\dagger U_3U_4^\dagger)-2tr(U_4U_3^\dagger U_2U_1^\dagger)\overset{3g}{=}-B^-_{133}+B^-_{233}+B^-_{144}-B^-_{244}+B^-_{344}+B^-_{122}$$
$$-((U_1-U_2)(U_2-U_3)(U_3-U_4)+(U_3-U_4)(U_2-U_3)(U_1-U_2))\cdot E\cdot E$$
$$\overset{3g}{=}-B^-_{133}+B^-_{233}+B^-_{144}-B^-_{244}+B^-_{344}+B^-_{122}-2(U_1-U_2)\cdot(U_2-U_3)\cdot(U_3-U_4)$$
$$\overset{3g}{=}-B^-_{133}+B^-_{233}+B^-_{144}-B^-_{244}+B^-_{344}+B^-_{122}$$
$$-(U_1-U_2)\cdot(U_2-U_3)\cdot(U_3-U_4)+(U_1^\dagger-U_2^\dagger)\cdot(U_2^\dagger-U_3^\dagger)\cdot(U_3^\dagger-U_4^\dagger)$$
$$=B^-_{144}+B^-_{322}-B^-_{433}-B^-_{211}+B^-_{124}+B^-_{234}-B^-_{123}-B^-_{134}. \tag{D.3}$$

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