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Abstract

It is shown that in the next-to-leading approximation of $N = 4$ SUSY the BFKL equation for two-gluon composite states in the adjoint representation of the gauge group can be reduced to a form which is invariant under Möbius transformation in the momentum space. The corresponding similarity transformation of its integral kernel is constructed in an explicit way.

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1 Introduction

The BFKL equation for the Pomeron wave function in the color singlet representation is well known [1]. In particular for the description of total cross-sections at high energies $\sqrt{s}$ its simple form at vanishing momentum transfers $|q| = \sqrt{-t} = 0$ is used. The integral kernel of this equation was calculated in the next-to-leading order (NLO) for QCD [2] and for $N = 4$ SUSY [3]. But the BFKL approach is applicable also for arbitrary t-channel color states constructed from two gluons. The corresponding NLO kernels at momentum transfers $q \neq 0$ are known both in QCD [4] and in $N = 4$ SUSY [5]. For the phenomenological applications the most important cases are the color singlet states constructed from two or several reggeized gluons. The corresponding Regge poles appear in the amplitudes having the antisymmetric adjoint representation (the f-coupling) in the t-channel. The concept of gluon reggeization was formulated on the base of the fixed-order calculations [1] and was checked in the leading logarithmic approximation (LLA) with the use of the so-called bootstrap equations [6], which follow from the compatibility of the multi-Regge form of production amplitudes with the s-channel unitarity. Later the bootstrap equations were constructed in the NLO [7]. Now the fulfillment of the corresponding relations in the NLO is proved (see [8] and references therein). Note that one can use the effective action for the calculation of the gluon Regge trajectory and the reggeon couplings in upper orders of perturbation theory [9].

There are at least two other reasons for the significance of the BFKL kernel in the adjoint representation. Its first application is related to the Bartels-Kwiecinski-Praszalowicz (BKP) equation [10]. This equation describes bound states of several reggeized gluons, in particular the Odderon which is a C-odd three-gluon state. In the last case, the pair-wise part of the NLO BKP kernel contains the NLO BFKL kernel for the symmetric adjoint representation (the d-coupling) [11]. Note that the difference between the symmetric and anti-symmetric representations appears only in NLO and even in this case the corresponding kernels coincide in the limit of large number of colors or provided that all particles in the action belong to the adjoint representation of the gauge group, as in the $N = 4$ SUSY. Another application of the BFKL approach was suggested in the framework of $N = 4$ SUSY to verify and generalize the Bern-Dixon-Smirnov (BDS) ansatz [12] for the production amplitudes with maximal helicity violation in the limit of large number of colors. It gave a possibility to find the high-energy behavior of the remainder function for the BDS ansatz [13, 14, 15] and to establish the relation of this problem with an integrable open spin chain [16].

The remainder function for the 6-point scattering amplitude in the kinematical regions containing the Mandelstam cut contribution was calculated recently in NLO [17], reproducing the two-loop expression for this function suggested by L. Dixon and collaborators [18]. It was done using the NLO BFKL kernel for the adjoint representation of the gauge group with subtracted gluon trajectory depending on the total momentum transfer $\vec{q}$. The eigenvalues of the kernel at large $\vec{q}^{\,2}$ were found and the BFKL equation was solved assuming the Möbius invariance of the modified kernel $\hat{\mathcal{K}}$ (with an omitted factor $\vec{q}^{\,2}$) in the two-dimensional transverse momentum space. The existence of the Möbius invariant kernel $\hat{\mathcal{K}}$ follows from general arguments related to the dual conformal invariance of the remainder function. However, the known NLO expression for $\hat{\mathcal{K}}$ obtained in the standard approach is not conformal invariant. In principle, this does not contradict the above assumption, because the NLO kernel is scheme-dependent. But for the verification of our assumption a similarity transformation reducing the standard kernel to the Möbius invariant expression should exist. Below we construct such transformation in the momentum space explicitly.

2 Standard and Möbius invariant forms of the kernel

The modified BFKL kernel $\hat{\mathcal{K}}$ for the two-gluon composite state in the antisymmetric adjoint representation is obtained by subtracting the gluon trajectory depending on $\vec{q}$ and extracting the factor $\vec{q}^{\,2}$ from its initial form:

$$\vec{q}^{\,2}\, K(\vec{q}_1, \vec{q}_1^{\,\prime}; \vec{q}) = \delta^{(2)}(\vec{q}_1 - \vec{q}_1^{\,\prime})\vec{q}_1^{\,2}\vec{q}_2^{\,2}\left(\omega_g(-\vec{q}_1^{\,2}) + \omega_g(-\vec{q}_2^{\,2}) - \omega_g(-\vec{q}^{\,2})\right)$$
$$+K_r(\vec{q}_1, \vec{q}_1^{\,\prime}; \vec{q})\,, \tag{1}$$

where $\vec{q}_i$ and $\vec{q}_i^{\,\prime}$ for $i = 1, 2$ are two reggeon momenta, $\vec{q} = \vec{q}_1 + \vec{q}_2 = \vec{q}_1^{\,\prime} + \vec{q}_2^{\,\prime}$, ω_g is the trajectory and K_r is the contribution coming from real particle production. The term $\omega_g(-\vec{q}^{\,2})$ is subtracted because the modified kernel is used for finding the high-energy behavior of the conformal invariant remainder function to the BDS ansatz containing the corresponding Regge factor.

Taken separately, the trajectories and the real part are infrared-divergent. It is known that the divergences are canceled in the singlet (Pomeron) kernel. It occurs that they are canceled also in Eq. (1), just due to the subtraction of $\omega_g(-\vec{q}^{\,2})$. The cancelation takes place because of two important properties: first, the singular terms of the trajectories do not depend on reggeon momenta, and second, the singular contribution to the real part of the kernel for the adjoint representation is two times smaller than the similar contribution to the Pomeron kernel.

Therefore, the modified kernel can be written in the form (cf. [17])

$$K(\vec{q}_1, \vec{q}_1^{\,\prime}; \vec{q}) = \frac{\vec{q}_1^{\,2}\vec{q}_2^{\,2}}{2\vec{q}^{\,2}}\delta^{(2)}(\vec{q}_1 - \vec{q}_1^{\,\prime})\left(\omega_g(-\vec{q}_1^{\,2}) + \omega_g(-\vec{q}_2^{\,2}) - 2\omega_g(-\vec{q}^{\,2})\right)$$

$$+K^f(\vec{q}_1, \vec{q}_1^{\,\prime}; \vec{q}), \qquad (2)$$

where both terms are infrared-finite. In particular,

$$\omega_g(-\vec{q}_1^{\,2}) + \omega_g(-\vec{q}_2^{\,2}) - 2\omega_g(-\vec{q}^{\,2}) = -\frac{\alpha_s N_c}{2\pi}\left(1 - \zeta(2)\,\frac{\alpha_s N_c}{2\pi}\right)\ln\left(\frac{\vec{q}_1^{\,2}\vec{q}_2^{\,2}}{\vec{q}^{\,4}}\right). \qquad (3)$$

Note that for the gauge coupling constant we use the dimensional reduction instead of the dimensional regularization which violates supersymmetry. This corresponds to the finite charge renormalization

$$\alpha_s(\mu) \to \alpha_s(\mu)\left(1 - \frac{\alpha_s(\mu)N_c}{12\pi}\right). \qquad (4)$$

One can write α_s instead of $\alpha_s(\mu)$, because in $N = 4$ the coupling constant is not running. Using the results of [11] for the contribution K^f and an integral representation for the difference of the trajectories obtained in Eq. (3), we present the modified kernel as follows:

$$K(\vec{q}_1, \vec{q}_1^{\,\prime}; \vec{q}) = K^B(\vec{q}_1, \vec{q}_1^{\,\prime}; \vec{q})\left(1 - \frac{\alpha_s N_c}{2\pi}\zeta(2)\right) + \delta^{(2)}(\vec{q}_1 - \vec{q}_1^{\,\prime})\frac{\vec{q}_1^{\,2}\vec{q}_2^{\,2}}{\vec{q}^{\,2}}\frac{\alpha_s^2 N_c^2}{4\pi^2}3\zeta(3)$$

$$+\frac{\alpha_s^2 N_c^2}{32\pi^3}R(\vec{q}_1, \vec{q}_1^{\,\prime}; \vec{q}), \qquad (5)$$

where K^B is the leading order kernel. It can be written in a form which is explicitly conformal invariant,

$$K^B(\vec{q}_1,\vec{q}_1\,';\vec{q}) = -\delta^{(2)}(\vec{q}_1-\vec{q}_1\,')\frac{\vec{q}_1^{\,2}\vec{q}_2^{\,2}}{\vec{q}^{\,2}}\frac{\alpha_s\,N_c}{4\pi^2}\int\frac{\vec{q}^{\,2}\,d^2l}{(\vec{q}_1-\vec{l})^2(\vec{q}_2+\vec{l})^2}$$

$$\times\left(\frac{\vec{q}_1^{\,2}(\vec{q}_2+\vec{l})^2+\vec{q}_2^{\,2}(\vec{q}_1-\vec{l})^2}{\vec{q}^{\,2}\vec{l}^{\,2}}-1\right)+\frac{\alpha_s\,N_c}{4\pi^2}\left(\frac{\vec{q}_1^{\,2}\vec{q}_2\,'^{\,2}+\vec{q}_1\,'^{\,2}\vec{q}_2^{\,2}}{\vec{q}^{\,2}\vec{k}^{\,2}}-1\right)\,. \quad (6)$$

Furthermore, the last term in the kernel (5) is written as

$$R(\vec{q}_1,\vec{q}_1\,',\vec{q}) = \frac{1}{2}\left(\ln\left(\frac{\vec{q}_1^{\,2}}{\vec{q}^{\,2}}\right)\ln\left(\frac{\vec{q}_2^{\,2}}{\vec{q}^{\,2}}\right)+\ln\left(\frac{\vec{q}_1\,'^{\,2}}{\vec{q}^{\,2}}\right)\ln\left(\frac{\vec{q}_2\,'^{2}}{\vec{q}^{\,2}}\right)+\ln^2\left(\frac{\vec{q}_1^{\,2}}{\vec{q}_1\,'^{\,2}}\right)\right)$$

$$-\frac{\vec{q}_1^{\,2}\vec{q}_2\,'^{\,2}+\vec{q}_2^{\,2}\vec{q}_1\,'^{\,2}}{\vec{q}^{\,2}\vec{k}^{\,2}}\ln^2\left(\frac{\vec{q}_1^{\,2}}{\vec{q}_1\,'^{\,2}}\right)-\frac{\vec{q}_1^{\,2}\vec{q}_2\,'^{\,2}-\vec{q}_2^{\,2}\vec{q}_1\,'^{\,2}}{2\vec{q}^{\,2}\vec{k}^{\,2}}\ln\left(\frac{\vec{q}_1^{\,2}}{\vec{q}_1\,'^{\,2}}\right)\ln\left(\frac{\vec{q}_1^{\,2}\vec{q}_1\,'^{\,2}}{\vec{k}^{\,4}}\right)$$

$$+4\frac{(\vec{k}\times\vec{q}_1)}{\vec{q}^{\,2}\vec{k}^{\,2}}\Big(\vec{k}^{\,2}(\vec{q}_1\times\vec{q}_2)-\vec{q}_1^{\;2}(\vec{k}\times\vec{q}_2)-\vec{q}_2^{\;2}(\vec{k}\times\vec{q}_1)\Big)\,I_{\vec{q}_1,-\vec{k}}$$

$$+\,(\vec{q}_1\leftrightarrow-\vec{q}_2,\;\;\vec{q}_1\,'\leftrightarrow-\vec{q}_2\,')\;\;. \quad (7)$$

Here $\vec{k}=\vec{q}_1-\vec{q}_1\,'=\vec{q}_2\,'-\vec{q}_2$, $(\vec{a}\times\vec{b})=a_xb_y-a_yb_x$ and

$$I_{\vec{p},\vec{q}}=\int_0^1\frac{dx}{(\vec{p}+x\vec{q})^2}\ln\left(\frac{\vec{p}^{\,2}}{x^2\vec{q}^{\,2}}\right)\;\;. \quad (8)$$

This quantity has the symmetry properties

$$I_{\vec{p},\vec{q}}=I_{-\vec{p},-\vec{q}}=I_{\vec{q},\vec{p}}=I_{\vec{p},-\vec{p}-\vec{q}}\;, \quad (9)$$

which are evident from the representation

$$I_{\vec{p},\vec{q}}=\int_0^1\int_0^1\int_0^1\frac{dx_1dx_2dx_3\delta(1-x_1-x_2-x_3)}{(\vec{p}^{\,2}x_1+\vec{q}^{\,2}x_2+(\vec{p}+\vec{q})^2x_3)(x_1x_2+x_1x_3+x_2x_3)}\;. \quad (10)$$

Other useful representations are

$$I_{\vec{p},\vec{q}}=\int_0^1\frac{dx}{\vec{p}^{\,2}(1-x)+\vec{q}^{\,2}x-(\vec{p}+\vec{q})^2x(1-x)}\ln\left(\frac{\vec{p}^{\,2}(1-x)+\vec{q}^{\,2}x}{(\vec{p}+\vec{q})^2x(1-x)}\right)$$

$$=\int_0^1dx\int_0^1dz\;\frac{1}{(\vec{p}+\vec{q})^2x(1-x)z+(\vec{q}^{\,2}(1-x)+\vec{p}^{\,2}x)(1-z)}\;. \quad (11)$$

In the kernel (5) the first two terms are conformal invariant (we remind that in our normalization the integration measure $\vec{q}^{\,2}d\vec{k}/(\vec{q}_1^{\,\prime\,2}\vec{q}_2^{\,\prime\,2})$ is Möbius invariant), but the contribution $R(\vec{q}_1, \vec{q}_1^{\,\prime}, \vec{q})$ violates this invariance. In [17] it was assumed that there is a conformal invariant representation of the kernel. Since its eigenvalues do not depend on the representation and on the total momentum transfer, they were found using the limit

$$|\vec{q}_1| \sim |\vec{q}_1^{\,\prime}| \ll |\vec{q}| \approx |\vec{q}_2| \approx |\vec{q}_2^{\,\prime}| \,. \tag{12}$$

In this limit the kernel (5) can be written as

$$K(z) = K^B(z)\left(1 - \frac{\alpha_s N_c}{2\pi}\zeta(2)\right) + \delta^{(2)}(1-z)\frac{\alpha_s^2 N_c^2}{4\pi^2}3\zeta(3) + \frac{\alpha_s^2 N_c^2}{32\pi^3}R(z)\,, \tag{13}$$

where $z = q_1/q_1^{\prime}$,

$$K^B(z) = \frac{\alpha_s N_c}{8\pi^2}\left(\frac{z+z^*}{|1-z|^2} - \delta^{(2)}(1-z)\int \frac{d\vec{l}}{|l|^2}\frac{l+l^*}{|1-l|^2}\right),$$

and

$$R(z) = \left(\frac{1}{2} - \frac{1+|z|^2}{|1-z|^2}\right)\ln^2|z|^2 - \frac{1-|z|^2}{2|1-z|^2}\ln|z|^2\,\ln\frac{|1-z|^4}{|z|^2}$$
$$+\left(\frac{1}{1-z} - \frac{1}{1-z^*}\right)(z-z^*)\int_0^1 \frac{dx}{|x-z|^2}\ln\frac{|z|^2}{x^2}\,. \tag{14}$$

Above we used the complex notations $r = x+iy$ and $r^* = x-iy$ for the two-dimensional vectors $\vec{r} = (x,y)$. Vice versa, two complex numbers z and z^* are equivalent to the vector $\vec{z}$ with the components $(z+z^*)/2$ and $(z-z^*)/(2i)$. Furthermore, $d\vec{r} = dxdy \equiv drdr^*/2\,,\quad \delta^{(2)}(\vec{r}) = 2\delta(r)\delta(r^*)$ and we define $\delta^{(2)}(z)$ in such a way that $\delta^{(2)}(z) = \delta(z)\delta(z^*)/2 = \delta^{(2)}(\vec{z})$.

Note that $K(z) = K(1/z)$. This property is evident for $K^B(z)$ and for the term with $\zeta(3)$. In the case of $R(z)$ it can be proved using the equality

$$\int_0^1 \frac{dx}{|x-z|^2}\ln\frac{|z|^2}{x^2} = \frac{1}{z-z^*}\left(2\int_0^1\frac{dx}{x}\ln\frac{1-xz^*}{1-xz} - \ln|z|^2\ln\frac{1-z^*}{1-z}\right) \tag{15}$$

and the relation

$$\mathrm{Li}_2(z) = -\mathrm{Li}_2\left(\frac{1}{z}\right) - \zeta(2) - \frac{1}{2}\ln^2(-z)\,. \tag{16}$$

The function $K(z)$ can be expanded in series over the complete set of functions

$$f_{\nu n}(z) = \frac{1}{\sqrt{2\pi^2}} |z|^{2i\nu} e^{in\phi} \,, \quad z = |z| e^{i\phi} \,, \tag{17}$$

with the orthogonality properties

$$\int \frac{d^2 z}{|z|^2} \, f^*_{\mu m}(z) \, f_{\nu n}(z) = \delta(\mu - \nu) \, \delta_{mn} \,. \tag{18}$$

This expansion looks as follows:

$$K(z) = \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\nu \, \omega(\nu, n) f_{\nu n}(z) \,, \tag{19}$$

where the eigenvalues $\omega(\nu, n)$ of the kernel are given by

$$\int \frac{d^2 z}{|z|^2} \, K(z) f^*_{\nu n}(z) = \omega(\nu, n) \,. \tag{20}$$

They were calculated in [17]. It was mentioned already that for the conformal invariant kernel the eigenvalues do not depend on the total momentum transfer. Therefore, the eigenvalues are known also for an arbitrary momentum transfer. As it is well known, an operator is completely defined by its eigenvalues and eigenfunctions. Formally, one can write

$$\hat{\mathcal{K}} = \sum_n \lambda_n |n\rangle \langle n| ,$$

where λ_n are the eigenvalues and $|n\rangle$ are the eigenfunctions normalized as

$$\langle n | n' \rangle = \delta_{nn'} \ .$$

Since we know that the eigenfunctions of the conformal invariant kernel [17] are

$$\phi_{\nu n}(q_1, q_2) = \frac{1}{\sqrt{2\pi^2}} \left(\frac{q_1}{q_2} \right)^{\frac{n}{2} + i\nu} \left(\frac{q_1^*}{q_2^*} \right)^{-\frac{n}{2} + i\nu} \,, \quad q_2 = q - q_1 \,, \tag{21}$$

with the normalization

$$\int \frac{\vec{q}^{\,2} d\vec{q}_1}{\vec{q}_1^{\,2} \vec{q}_2^{\,2}} \left(\phi_{\nu n}(q_1, q_2) \right)^* \phi_{\mu m}(q_1, q_2) = \int \frac{d^2 z}{|z|^2} \, f^*_{\mu m}(z) \, f_{\nu n}(z) = \delta(\mu - \nu) \, \delta_{mn} \,, \tag{22}$$

then, denoting the conformal kernel $\hat{\mathcal{K}}_c$, we can present it as follows:

$$K_c(\vec{q}_1, \vec{q}_1{}'; \vec{q}) = \sum_{n=-\infty}^{n=+\infty} \int d\nu \, \omega(\nu, n) \phi_{\nu n} \left(q_1, q_2\right) \left(\phi_{\nu n}(q_1', q_2')\right)^* \,. \tag{23}$$

But in fact there is no need to calculate this complicated expression. The matter is that, due to the Möbius invariance, the kernel $K_c(\vec{q}_1, \vec{q}_1{}'; \vec{q})$ can be written as $K(z)$ given in Eq. (13) with the argument $z = q_1 q_2'/(q_2 q_1')$.

Furthermore, if we denote

$$K(\vec{q}_1, \vec{q}_1{}'; \vec{q}) - K_c(\vec{q}_1, \vec{q}_1{}'; \vec{q}) = \frac{\alpha_s^2 N_c^2}{32\pi^3} \Delta(\vec{q}_1, \vec{q}_1{}'; \vec{q}) \,, \tag{24}$$

then, using the conformal symmetry of K^B and the term with $\zeta(3)$ in Eq. (5), one obtains from Eqs. (5) and (13)

$$\Delta(\vec{q}_1, \vec{q}_1{}'; \vec{q}) = R(\vec{q}_1, \vec{q}_1{}'; \vec{q}) - R(z) \,, \tag{25}$$

where $R(\vec{q}_1, \vec{q}_1{}'; \vec{q})$ is given in Eq. (7) and $R(z)$ in Eq. (14) with $z = q_1 q_2'/(q_2 q_1')$. Since $R(\vec{q}_1, \vec{q}_1{}'; \vec{q})$ is not conformal invariant, it cannot be written using the single variable z. Let us define the variables $z_i = q_1/q_1'$, $i = 1, 2$, $z_1/z_2 = z$. Then from Eqs. (7) and (14) one has

$$\Delta(\vec{q}_1, \vec{q}_1{}'; \vec{q}) = 2 \ln \left(\frac{|1-z|^2 |z_1|}{|1-z_1|^2 |z|} \right) \ln \left(\frac{|1-z|^2 |z_2|}{|1-z_2|^2 |z|} \right)$$

$$+ 6 \ln |z_1| \ln |z_2| - 8 \frac{1+|z|^2}{|1-z|^2} \ln |z_1| \ln |z_2|$$

$$+ 2 \frac{1-|z|^2}{|1-z|^2} \left(\ln |z_1| \ln \frac{|z_1|}{|1-z_1|^2} - \ln |z_2| \ln \frac{|z_2|}{|1-z_2|^2} - \ln |z| \ln \frac{|z|}{|1-z|^2} \right)$$

$$+ 2 \frac{z-z^*}{|1-z|^2} \left[\mathrm{Li}_2(z_1) - \mathrm{Li}_2(z_1^*) - \mathrm{Li}_2(z_2) + \mathrm{Li}_2(z_2^*) - \mathrm{Li}_2(z) + \mathrm{Li}_2(z^*) \right.$$

$$\left. + \ln |z_1| \ln \frac{1-z_1}{1-z_1^*} - \ln |z_2| \ln \frac{1-z_2}{1-z_2^*} - \ln |z| \ln \frac{1-z}{1-z^*} \right] . \tag{26}$$

Note that Δ is symmetric with respect to the exchange $1 \leftrightarrow 2$, i.e. $z_1 \leftrightarrow z_2, \ \ z \leftrightarrow 1/z$.

The dilogarithms entering Eq. (26) are not independent. Their number can be reduced using the relation

$$\mathrm{Li}_2(z_1/z_2) = \mathrm{Li}_2(z_1) + \mathrm{Li}_2(1/z_2) + \mathrm{Li}_2\left(\frac{z_1-1}{z_2-1}\right)$$
$$+\mathrm{Li}_2\left(\frac{z_1(z_2-1)}{z_2(z_1-1)}\right) + \frac{1}{2}\ln^2\left(\frac{(z_2-1)}{z_2(1-z_1)}\right) . \quad (27)$$

This gives

$$\Delta(\vec{q}_1, \vec{q}_1{}'; \vec{q}) = 2\ln\left(\frac{|1-z|^2|z_1|}{|1-z_1|^2|z|}\right)\ln\left(\frac{|1-z|^2|z_2|}{|1-z_2|^2|z|}\right)$$
$$+6\ln|z_1|\ln|z_2| - 8\frac{1+|z|^2}{|1-z|^2}\ln|z_1|\ln|z_2|$$
$$+2\frac{1-|z|^2}{|1-z|^2}\left(\ln|z_1|\ln\frac{|z_1|}{|1-z_1|^2} - \ln|z_2|\ln\frac{|z_2|}{|1-z_2|^2} - \ln|z|\ln\frac{|z|}{|1-z|^2}\right)$$
$$+2\frac{z-z^*}{|1-z|^2}\left[-\mathrm{Li}_2\left(\frac{(1-z_1)}{(1-z_2)}\right) + \mathrm{Li}_2^*\left(\frac{(1-z_1)}{(1-z_2)}\right) - \mathrm{Li}_2\left(\frac{(1-z_2)z}{(1-z_1)}\right)\right.$$
$$+\mathrm{Li}_2^*\left(\frac{(1-z_2)z}{(1-z_1)}\right) - \ln\left|\frac{(1-z_1)}{(1-z_2)}\right|\ln\left(\frac{(1-z)z_2(1-z_2^*)}{(1-z^*)z_2^*(1-z_2)}\right)$$
$$\left.-\ln\left|\frac{(1-z_2)z}{(1-z_1)}\right|\ln\left(\frac{(1-z)(1-z_1^*)}{(1-z^*)(1-z_1)}\right)\right] . \quad (28)$$

Note that

$$\frac{(1-z_1)}{(1-z_2)} = -\frac{q_2'}{q_1'}\,, \quad \frac{(1-z_2)z}{(1-z_1)} = -\frac{q_1}{q_2}\,, \quad \frac{(1-z)z_2}{(1-z_2)} = -\frac{q}{q_1'}\,, \quad \frac{(1-z)}{(1-z_1)} = \frac{q}{q_2}\,. \quad (29)$$

The symmetry with respect to the exchange $1 \leftrightarrow 2$, i.e. $z_1 \leftrightarrow z_2, \;\; z \leftrightarrow 1/z$, holds also here, although it is not so obvious as before.

3 Similarity transformation to the conformal form

If the kernels $\hat{\mathcal{K}}$ and $\hat{\mathcal{K}}_c$ coincide in the leading order and are related by a similarity transformation, there must exist an operator $\hat{\mathcal{O}}$ satisfying the commutation relation

$$[\hat{\mathcal{K}}^B, \hat{O}] = \left(\frac{\alpha_s}{2\pi}\right)^2 \frac{1}{8\pi}\hat{\Delta}\,. \quad (30)$$

One can find a formal expression for this operator allowing to construct the similarity transformation in perturbation theory. Indeed, it is enough to calculate the matrix element of the above commutation relation between the eigenfunctions (21) of the Born kernel with the corresponding eigenvalues $\omega^B_{\nu n}$ in the form

$$\left(\omega^B_{\nu' n'} - \omega^B_{\nu n}\right) \langle \nu' n' | \hat{\mathcal{O}} | \nu n \rangle = \left(\frac{\alpha_s}{2\pi}\right)^2 \frac{1}{8\pi} \langle \nu' n' | \hat{\Delta} | \nu n \rangle. \tag{31}$$

It can be seen from this equation that the solution $\hat{\mathcal{O}}$ exists only if the operator $\hat{\Delta}$ has vanishing matrix elements between states with the same eigenvalues. In this case, using the completeness of the functions $|\nu n\rangle$, we obtain

$$\hat{O} = \left(\frac{\alpha_s}{2\pi}\right)^2 \frac{1}{8\pi} \sum_{n,n'} \int d\nu \int d\nu' \frac{|\nu' n'\rangle \langle \nu' n' | \hat{\Delta} | \nu n \rangle \langle \nu n |}{\omega^B_{\nu' n'} - \omega^B_{\nu n}} \tag{32}$$

and

$$\langle \vec{q}_1, \vec{q}_2 | \hat{\mathcal{O}} | \vec{q}_1^{\,\prime}, \vec{q}_2^{\,\prime} \rangle = \left(\frac{\alpha_s}{2\pi}\right)^2 \frac{1}{8\pi} \sum_{n,n'} \int d\nu \int d\nu' \frac{\langle \vec{q}_1, \vec{q}_2 | \nu' n' \rangle \langle \nu' n' | \hat{\Delta} | \nu n \rangle \langle \nu n | \vec{q}_1^{\,\prime}, \vec{q}_2^{\,\prime} \rangle}{\omega^B_{\nu' n'} - \omega^B_{\nu n}}. \tag{33}$$

Since the kernel $\hat{\Delta}$ is known in the momentum space (see (26)), we can transform it into the (n, ν) representation,

$$\langle \nu n | \hat{\Delta} | \nu' n' \rangle = \int \frac{\vec{q}^{\,2} d\vec{q}_1}{\vec{q}_1^{\,2} (\vec{q} - \vec{q}_1)^2} \int \frac{\vec{q}^{\,2} d\vec{q}_1^{\,\prime}}{\vec{q}_1^{\,\prime\,2} (\vec{q} - \vec{q}_1^{\,\prime})^2} \langle \nu' n' | \vec{q}_1^{\,\prime}, \vec{q}_2^{\,\prime} \rangle \Delta(\vec{q}_1, \vec{q}_1^{\,\prime}; \vec{q}) \langle \vec{q}_1, \vec{q}_2 | \nu n \rangle \tag{34}$$

using the known eigenfunctions in the momentum space (21), which allows to find the matrix element $\langle \vec{q}_1, \vec{q}_2 | \hat{\mathcal{O}} | \vec{q}_1^{\,\prime}, \vec{q}_2^{\,\prime} \rangle$.

The eigenfunctions (21) entering in (34) depend on $r = q_1/q_2$ and $r' = q_1'/q_2'$; therefore it is convenient to express $\Delta(\vec{q}_1, \vec{q}_1^{\,\prime}; \vec{q})$ (28) in terms of r, r' and $z = r/r'$. In these variables we have

$$\Delta = 2 \ln^2 \frac{|1+r|^2}{|r|} + 2 \ln^2 \frac{|1+r'|^2}{|r'|} - 2 \ln \frac{|1+r|^2}{|r|} \ln \frac{|1+r'|^2}{|r'|}$$
$$-2 \ln^2 |r| - 2 \ln^2 |r'| + 2 \ln |r| \ln |r'|$$
$$-2 \frac{1+|z|^2}{|1-z|^2} \left(\ln^2 \frac{|1+r|^2}{|r|} + \ln^2 \frac{|1+r'|^2}{|r'|} - 2 \ln \frac{|1+r|^2}{|r|} \ln \frac{|1+r'|^2}{|r'|} - \ln^2 |z| \right)$$
$$+2 \frac{1-|z|^2}{|1-z|^2} \left(\ln |r| \ln \frac{|1+r'|^2}{|r'|} - \ln |r'| \ln \frac{|1+r|^2}{|r|} \right)$$

$$+2\frac{z-z^*}{|1-z|^2}\Big[-\mathrm{Li}_2(-r)+\mathrm{Li}_2^*(-r)+\mathrm{Li}_2(-r')-\mathrm{Li}_2^*(-r')$$
$$-\ln|r|\ln\left(\frac{1+r}{1+r^*}\right)+\ln|r'|\ln\left(\frac{1+r'}{1+r'^*}\right)\Big]. \quad (35)$$

In terms of the variables r, r' the symmetry with respect to the exchange $1 \leftrightarrow 2$ is equivalent to the symmetry of the above expression under the transformations $r \leftrightarrow 1/r,\;\; r' \leftrightarrow 1/r',\;\; z \leftrightarrow 1/z$.

Note that

$$2\ln^2\frac{|1+r|^2}{|r|}+2\ln^2\frac{|1+r'|^2}{|r'|}-2\ln\frac{|1+r|^2}{|r|}\ln\frac{|1+r'|^2}{|r'|}$$
$$-2\ln^2|r|+2\ln^2|r'|-2\ln|r|\ln|r'|$$
$$=2\ln\frac{|1+r|^2}{|1+r'|}\ln\frac{|1+r|^2}{|r|^2}+2\ln\frac{|1+r'|^2}{|1+r|}\ln\frac{|1+r'|^2}{|r'|^2}\,, \quad (36)$$

which demonstrates the absence of singularities at $r = 0$. Analogously,

$$\ln^2\frac{|1+r|^2}{|r|}+\ln^2\frac{|1+r'|^2}{|r'|}-2\ln\frac{|1+r|^2}{|r|}\ln\frac{|1+r'|^2}{|r'|}-\ln^2|z|$$
$$=\ln\frac{|1+r|^2}{|1+r'|^2}\ln\frac{|1+r|^2{r'}^2}{|1+r'|^2r^2}\,. \quad (37)$$

In order to obtain $\langle\vec{q}_1,\vec{q}_2|\hat{\mathcal{O}}|\vec{q}_1{}',\vec{q}_2{}'\rangle$ by this method one has to perform complicated calculations. The first difficulty appears in calculating the matrix elements $\langle\nu'n'|\hat{\Delta}|\nu n\rangle$. Here the main problem arises from the term in Δ proportional to the expression

$$\frac{1+|z|^2}{|1-z|^2}\ln\frac{|1+r|^2}{|r|}\ln\frac{|1+r'|^2}{|r'|},$$

because the corresponding integral is not factorized. It leads to infinite double sums over poles in the complex planes with positions depending on n, ν and n', ν'. At the end one should calculate the complicated double sum and double integral in Eq. (33). But the final result turns out to be quite simple. Moreover, it can be guessed, as it is shown in the next section.

4 Explicit form of the similarity transformation

To diminish the search area we have to use all possible restrictions implied on $\hat{\mathcal{O}}$. Important restrictions follow from symmetries and the gauge invariance of $\hat{\Delta}$ and $\hat{\mathcal{K}}_r^B$. Due to the symmetries of $\hat{\Delta}$ and $\hat{\mathcal{K}}^B$ we should look for $\hat{\mathcal{O}}$ among operators symmetric under the interchange $1 \leftrightarrow 2$ and antisymmetric under transposition. The last property excludes diagonal terms (proportional to $\delta(\vec{q}_1 - \vec{q}_1{}')$ in momentum space). The non-diagonal part can be taken gauge invariant.

There is only one possibility (up to a coefficient) for such operator without logarithms, and it is just $\hat{\mathcal{K}}_r^B$. However, the operator which we are looking for must contain logarithms, as it follows from Eqs. (31) and (35). This enlarges the number of such operators drastically. But there is one additional argument. It seems quite natural that the conformal kernel can be obtained by modification of the subtraction procedure used in the definition of the standard kernel [19] for separation of leading and next-to-leading contributions. In this case the operator $\hat{\mathcal{O}}$ must be proportional to $\hat{\mathcal{K}}_r^B$. Using the requirement of antisymmetry under transposition, we come to the conclusion that the most attractive candidate for $\hat{\mathcal{O}}$ is

$$\hat{\mathcal{O}} = C\left[\ln\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right), \hat{\mathcal{K}}_r^B\right], \tag{38}$$

where C is some coefficient.

Let us show that indeed the required operator has the form (38) with $C = 1/4$. In the momentum space it looks as

$$O(\vec{q}_1, \vec{q}_1{}'; \vec{q}\,) \;=\; \frac{\alpha_s N_c}{16\pi^2}\left(\frac{\vec{q}_1^{\,2}\vec{q}_2{}'^{\,2} + \vec{q}_1{}'^{\,2}\vec{q}_2^{\,2}}{\vec{k}^{\,2}} - \vec{q}^{\,2}\right)\ln\left(\frac{\vec{q}_1^{\,2}\vec{q}_2^{\,2}}{\vec{q}_1{}'^{\,2}\vec{q}_2{}'^{\,2}}\right). \tag{39}$$

To obtain $\Delta(\vec{q}_1, \vec{q}_1{}'; \vec{q}\,)$ in vector denotations from Eq. (35) the following relations are useful:

$$2\Big[-\mathrm{Li}_2(-r) + \mathrm{Li}_2^*(-r) + \mathrm{Li}_2(-r') - \mathrm{Li}_2^*(-r')$$
$$- \ln|r|\ln\left(\frac{1+r}{1+r^*}\right) + \ln|r'|\ln\left(\frac{1+r'}{1+r'^*}\right)\Big]$$
$$= (q_1 q_2^* - q_1^* q_2) I_{\vec{q}_1, \vec{q}_2} + (q_2' q_1'^* - q_2'^* q_1') I_{\vec{q}_1{}', \vec{q}_2{}'}\,, \quad ab^* - a^*b = -2i[\vec{a}\times\vec{b}\,]\,, \tag{40}$$
$$\frac{z - z^*}{|1-z|^2} = \frac{2i}{\vec{k}^{\,2}\vec{q}^{\,2}}\left(\vec{k}^{\,2}[\vec{q}_1\times\vec{q}_2] - \vec{q}_1^{\,2}[\vec{k}\times\vec{q}_2] - \vec{q}_2^{\,2}[\vec{k}\times\vec{q}_1]\right)$$

$$= \frac{-2i}{\vec{k}^{\,2}\vec{q}^{\,2}}\left(\vec{k}^{\,2}[\vec{q}_1^{\,\prime}\times\vec{q}_2^{\,\prime}]+\vec{q}_1^{\,\prime\,2}[\vec{k}\times\vec{q}_2^{\,\prime}]+\vec{q}_2^{\,\prime\,2}[\vec{k}\times\vec{q}_1^{\,\prime}]\right) . \qquad (41)$$

The last equality follows from antisymmetry with respect to $\vec{q}_i \leftrightarrow -\vec{q}_i^{\,\prime}$.

Using these relations, it is easy to obtain

$$\Delta(\vec{q}_1,\vec{q}_1^{\,\prime};\vec{q}) = \ln\frac{\vec{q}_1^{\,2}}{\vec{q}^{\,2}}\ln\frac{\vec{q}_2^{\,2}}{\vec{q}^{\,2}}+\ln\frac{\vec{q}_1^{\,\prime\,2}}{\vec{q}^{\,2}}\ln\frac{\vec{q}_2^{\,\prime\,2}}{\vec{q}^{\,2}}+\ln\frac{\vec{q}_1^{\,2}}{\vec{q}_1^{\,\prime\,2}}\ln\frac{\vec{q}_2^{\,2}}{\vec{q}_2^{\,\prime\,2}}$$
$$-2\frac{\vec{q}_1^{\,2}\vec{q}_2^{\,\prime\,2}+\vec{q}_2^{\,2}\vec{q}_1^{\,\prime\,2}}{\vec{k}^{\,2}\vec{q}^{\,2}}\ln\frac{\vec{q}_1^{\,2}}{\vec{q}_1^{\,\prime\,2}}\ln\frac{\vec{q}_2^{\,2}}{\vec{q}_2^{\,\prime\,2}}$$
$$+\frac{\vec{q}_1^{\,2}\vec{q}_2^{\,\prime\,2}-\vec{q}_2^{\,2}\vec{q}_1^{\,\prime\,2}}{\vec{k}^{\,2}\vec{q}^{\,2}}\left(\ln\frac{\vec{q}_1^{\,2}}{\vec{q}^{\,2}}\ln\frac{\vec{q}_2^{\,\prime\,2}}{\vec{q}^{\,2}}-\ln\frac{\vec{q}_2^{\,2}}{\vec{q}^{\,2}}\ln\frac{\vec{q}_1^{\,\prime\,2}}{\vec{q}^{\,2}}\right)$$
$$+\frac{4}{\vec{q}^{\,2}\vec{k}^{\,2}}\left(\vec{k}^{\,2}[\vec{q}_1\times\vec{q}_2]-\vec{q}_1^{\,2}[\vec{k}\times\vec{q}_2]-\vec{q}_2^{\,2}[\vec{k}\times\vec{q}_1]\right)$$
$$\times\left([\vec{q}_1\times\vec{q}_2]I_{\vec{q}_1,\vec{q}_2}-[\vec{q}_1^{\,\prime}\times\vec{q}_2^{\,\prime}]I_{\vec{q}_1^{\,\prime},\vec{q}_2^{\,\prime}}\right) . \qquad (42)$$

Important properties of Δ are its symmetries with respect to the exchanges $\vec{q}_1 \leftrightarrow -\vec{q}_2$, $\vec{q}_1^{\,\prime} \leftrightarrow -\vec{q}_2^{\,\prime}$ and $\vec{q}_i \leftrightarrow -\vec{q}_i$, as well as the gauge invariance (vanishing at zero momentum of each reggeon), which are easily seen from this representation.

Since the kernel $\hat{\mathcal{K}}^B$ contains real and virtual parts, the commutator

$$\hat{\mathcal{D}} = [\hat{\mathcal{K}}^B,\hat{\mathcal{O}}] = \frac{1}{4}\left[\hat{\mathcal{K}}^B,\left[\ln\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right),\hat{\mathcal{K}}_r^B\right]\right] \qquad (43)$$

is naturally separated into two pieces. One is $\hat{\mathcal{D}}_v = [\hat{\omega}_{g_1}+\hat{\omega}_{g_2},\hat{\mathcal{O}}]$ and gives in the momentum space

$$D_v(\vec{q}_1,\vec{q}_2;\,\vec{k}) = \left(\omega_g(-\vec{q}_1^{\,2})+\omega_g(-\vec{q}_2^{\,2})-\omega_g(-\vec{q}_1^{\,\prime\,2})-\omega_g(-\vec{q}_2^{\,\prime\,2})\right)O(\vec{q}_1,\vec{q}_1^{\,\prime};\vec{q})$$
$$= -\frac{\alpha_s^2N_c^2}{32\pi^3}\left(\frac{\vec{q}_1^{\,2}\vec{q}_2^{\,\prime\,2}+\vec{q}_1^{\,\prime\,2}\vec{q}_2^{\,2}}{\vec{k}^{\,2}}-\vec{q}^{\,2}\right)\ln^2\left(\frac{\vec{q}_1^{\,2}\vec{q}_2^{\,2}}{\vec{q}_1^{\,\prime\,2}\vec{q}_2^{\,\prime\,2}}\right) . \qquad (44)$$

The piece $\hat{\mathcal{D}}_r = [\hat{\mathcal{K}}_r^B,\hat{\mathcal{O}}]$ can be written in the momentum space as

$$D_r(\vec{q}_1,\vec{q}_2;\,\vec{k}) = frac\vec{q}_1^{\,2}\vec{q}_2^{\,2}4\int d\vec{p}\,\frac{K_r(\vec{q}_1,\vec{q}_2;\,\vec{q}_1-\vec{p})}{\vec{q}_1^{\,2}\vec{q}_2^{\,2}}\frac{K_r(\vec{p},\vec{q}-\vec{p};\,\vec{p}-\vec{q}_1^{\,\prime})}{\vec{p}^{\,2}(\vec{q}-\vec{p})^2}$$
$$\times\ln\left(\frac{(\vec{p}^{\,2})^2}{\vec{q}_1^{\,2}\vec{q}_1^{\,\prime\,2}}\frac{((\vec{q}-\vec{p})^2)^2}{\vec{q}_2^{\,2}\vec{q}_2^{\,\prime\,2}}\right) . \qquad (45)$$

A convenient way to calculate this integral is to use complex variables for the two-dimensional vectors and to perform the pole expansion of the integrand. We have

$$\frac{K_r(\vec{q}_1,\vec{q}_2;\ \vec{q}_1-\vec{p})}{\vec{q}_1^{\,2}\vec{q}_2^{\,2}}\frac{K_r(\vec{p},\vec{q}-\vec{p};\ \vec{p}-\vec{q}_1^{\,\prime})}{\vec{p}^{\,2}(\vec{q}-\vec{p})^2}$$

$$= \left(\frac{\alpha_s N_c}{4\pi^2}\right)^2\Bigg[\left(\frac{1}{(\vec{q}_1^{\,\prime}-\vec{p})^2}+\frac{1}{(\vec{q}_1-\vec{p})^2}\right)\left(\frac{\vec{q}_1^{\,2}\vec{q}_2^{\,\prime\,2}+\vec{q}_1^{\,\prime\,2}\vec{q}_2^{\,2}}{\vec{k}^{\,2}\vec{q}_1^{\,2}\vec{q}_2^{\,2}}-\frac{\vec{q}^{\,2}}{\vec{q}_1^{\,2}\vec{q}_2^{\,2}}\right)$$

$$-\frac{1}{(q_1-p)(q_1^{\prime *}-p^*)}\frac{q_1^\prime q^*}{\vec{q}_1^{\,2}kq_2^*}-\frac{1}{(q_1-p)p^*}\frac{q_2^\prime q^*}{\vec{q}_2^{\,2}kq_1^*}+\frac{1}{(q_1^\prime-p)(q^*-p^*)}\frac{q_2^\prime q^*}{\vec{q}_2^{\,2}kq_1^*}$$

$$+\frac{1}{(q_1^\prime-p)p^*}\frac{q_1^\prime q^*}{\vec{q}_1^{\,2}kq_2^*}-\frac{1}{(q_1-p)(q_1^{\prime *}-p^*)}\frac{1}{\vec{k}^{\,2}}\left|\frac{q_1^\prime}{q_1}+\frac{q_2^\prime}{q_2}\right|^2+\frac{1}{p(q^*-p^*)}\frac{\vec{q}^{\,2}}{\vec{q}_2^{\,2}\vec{q}_2^{\,2}}+\text{c.c.}\Bigg]. \tag{46}$$

Taken separately, each term in this expansion gives an ultraviolet-divergent integral. Of course, the divergences cancel in their sum. Introducing the intermediate cutoff $\vec{p}^{\,2}\le\Lambda^2$, one has

$$\int\frac{d\vec{p}}{\pi}\ln\left(\frac{(\vec{p}^{\,2})^2}{\vec{q}_1^{\,2}\vec{q}_1^{\,\prime\,2}}\right)\left(\frac{1}{(\vec{q}_1^{\,\prime}-\vec{p})^2}+\frac{1}{(\vec{q}_1-\vec{p})^2}\right)\theta(\Lambda^2-\vec{p}^{\,2})=\ln^2\left(\frac{\Lambda^2}{\vec{q}_1^{\,2}}\right)+\ln^2\left(\frac{\Lambda^2}{\vec{q}_1^{\,\prime\,2}}\right),$$

$$\int\frac{d\vec{p}}{\pi}\frac{2}{(a-p)(b^*-p^*)}\ln\left(\frac{\vec{p}^{\,2}}{\mu^2}\right)\theta(\Lambda^2-\vec{p}^{\,2})=\ln\left(\frac{\Lambda^2}{(\vec{a}-\vec{b})^2}\right)\ln\left(\frac{\Lambda^2(\vec{a}-\vec{b})^2}{\mu^4}\right)$$

$$+\ln\left(\frac{(\vec{a}-\vec{b})^2}{\vec{b}^{\,2}}\right)\ln\left(\frac{(\vec{a}-\vec{b})^2}{\vec{a}^{\,2}}\right)+(ab^*-a^*b)I_{\vec{a},-\vec{b}}\ . \tag{47}$$

Using also equalities

$$\frac{q_1^\prime q^*}{\vec{q}_1^{\,2}kq_2^*}+\text{c.c.}=\frac{\vec{q}_1^{\,2}\vec{q}_2^{\,\prime\,2}-\vec{q}_2^{\,2}\vec{q}_1^{\,\prime\,2}-\vec{q}^{\,2}\vec{k}^{\,2}}{\vec{q}_1^{\,2}\vec{q}_2^{\,2}\vec{k}^{\,2}}\ ,$$

$$\frac{q_2^\prime q^*}{\vec{q}_2^{\,2}kq_1^*}+\text{c.c.}=\frac{\vec{q}_1^{\,2}\vec{q}_2^{\,\prime\,2}-\vec{q}_2^{\,2}\vec{q}_1^{\,\prime\,2}+\vec{q}^{\,2}\vec{k}^{\,2}}{\vec{q}_1^{\,2}\vec{q}_2^{\,2}\vec{k}^{\,2}}\ ,$$

$$\frac{1}{\vec{k}^{\,2}}\left|\frac{q_1^\prime}{q_1}+\frac{q_2^\prime}{q_2}\right|^2=\frac{2(\vec{q}_1^{\,2}\vec{q}_2^{\,\prime\,2}+\vec{q}_2^{\,2}\vec{q}_1^{\,\prime\,2})-\vec{q}^{\,2}\vec{k}^{\,2}}{\vec{q}_1^{\,2}\vec{q}_2^{\,2}\vec{k}^{\,2}}\ , \tag{48}$$

we obtain

$$D_r(\vec{q}_1, \vec{q}_2;\ \vec{k}) = \frac{\alpha_s^2 N_c^2}{32\pi^3}\vec{q}^{\,2}\Bigg[\ln\frac{\vec{q}_1^{\,2}}{\vec{q}^{\,2}}\ln\frac{\vec{q}_2^{\,2}}{\vec{q}^{\,2}} + \ln\frac{\vec{q}_1^{\,\prime\,2}}{\vec{q}^{\,2}}\ln\frac{\vec{q}_2^{\,\prime\,2}}{\vec{q}^{\,2}} - \ln\frac{\vec{q}_1^{\,2}}{\vec{q}_1^{\,\prime\,2}}\ln\frac{\vec{q}_2^{\,2}}{\vec{q}_2^{\,\prime\,2}}$$

$$+\left(\frac{\vec{q}_1^{\,2}\vec{q}_2^{\,\prime\,2} + \vec{q}_2^{\,2}\vec{q}_1^{\,\prime\,2}}{\vec{k}^{\,2}\vec{q}^{\,2}} - 1\right)\left(\ln^2\frac{\vec{q}_1^{\,2}}{\vec{q}_1^{\,\prime\,2}} + \ln^2\frac{\vec{q}_2^{\,2}}{\vec{q}_2^{\,\prime\,2}}\right)$$

$$+\frac{\vec{q}_1^{\,2}\vec{q}_2^{\,\prime\,2} - \vec{q}_2^{\,2}\vec{q}_1^{\,\prime\,2}}{\vec{k}^{\,2}\vec{q}^{\,2}}\left(\ln\frac{\vec{q}_1^{\,2}}{\vec{q}^{\,2}}\ln\frac{\vec{q}_2^{\,\prime\,2}}{\vec{q}^{\,2}} - \ln\frac{\vec{q}_2^{\,2}}{\vec{q}^{\,2}}\ln\frac{\vec{q}_1^{\,\prime\,2}}{\vec{q}^{\,2}}\right)$$

$$+\frac{4}{\vec{q}^{\,2}\vec{k}^{\,2}}\left(\vec{k}^{\,2}[\vec{q}_1\times\vec{q}_2] - \vec{q}_1^{\,2}[\vec{k}\times\vec{q}_2] - \vec{q}_2^{\,2}[\vec{k}\times\vec{q}_1]\right)$$

$$\times\left([\vec{q}_1\times\vec{q}_2]I_{\vec{q}_1,\vec{q}_2} - [\vec{q}_1^{\,\prime}\times\vec{q}_2^{\,\prime}]I_{\vec{q}_1^{\,\prime},\vec{q}_2^{\,\prime}}\right)\Bigg]\,. \tag{49}$$

The total commutator $\hat{\mathcal{D}} = [\hat{\mathcal{K}}^B, \hat{\mathcal{O}}_t]$ is defined in the momentum space by the sum of the two pieces given in Eqs. (44) and (49):

$$D(\vec{q}_1, \vec{q}_2;\ \vec{k}) = \frac{\alpha_s^2 N_c^2}{32\pi^3}C\vec{q}^{\,2}\Bigg[\ln\frac{\vec{q}_1^{\,2}}{\vec{q}^{\,2}}\ln\frac{\vec{q}_2^{\,2}}{\vec{q}^{\,2}} + \ln\frac{\vec{q}_1^{\,\prime\,2}}{\vec{q}^{\,2}}\ln\frac{\vec{q}_2^{\,\prime\,2}}{\vec{q}^{\,2}} + \ln\frac{\vec{q}_1^{\,2}}{\vec{q}_1^{\,\prime\,2}}\ln\frac{\vec{q}_2^{\,2}}{\vec{q}_2^{\,\prime\,2}}$$

$$-2\frac{\vec{q}_1^{\,2}\vec{q}_2^{\,\prime\,2} + \vec{q}_2^{\,2}\vec{q}_1^{\,\prime\,2}}{\vec{k}^{\,2}\vec{q}^{\,2}}\ln\frac{\vec{q}_1^{\,2}}{\vec{q}_1^{\,\prime\,2}}\ln\frac{\vec{q}_2^{\,2}}{\vec{q}_2^{\,\prime\,2}}$$

$$+\frac{\vec{q}_1^{\,2}\vec{q}_2^{\,\prime\,2} - \vec{q}_2^{\,2}\vec{q}_1^{\,\prime\,2}}{\vec{k}^{\,2}\vec{q}^{\,2}}\left(\ln\frac{\vec{q}_1^{\,2}}{\vec{q}^{\,2}}\ln\frac{\vec{q}_2^{\,\prime\,2}}{\vec{q}^{\,2}} - \ln\frac{\vec{q}_2^{\,2}}{\vec{q}^{\,2}}\ln\frac{\vec{q}_1^{\,\prime\,2}}{\vec{q}^{\,2}}\right)$$

$$+\frac{4}{\vec{q}^{\,2}\vec{k}^{\,2}}\left(\vec{k}^{\,2}[\vec{q}_1\times\vec{q}_2] - \vec{q}_1^{\,2}[\vec{k}\times\vec{q}_2] - \vec{q}_2^{\,2}[\vec{k}\times\vec{q}_1]\right)$$

$$\times\left([\vec{q}_1\times\vec{q}_2]I_{\vec{q}_1,\vec{q}_2} - [\vec{q}_1^{\,\prime}\times\vec{q}_2^{\,\prime}]I_{\vec{q}_1^{\,\prime},\vec{q}_2^{\,\prime}}\right)\Bigg]\,. \tag{50}$$

Comparing Eq. (50) with Eq. (42) and taking into account Eq. (43), one sees that indeed Eq. (30) is fulfilled, if $\hat{\mathcal{O}}$ is given by (38) with $C = 1/4$. Using (24), we conclude that

$$\hat{\mathcal{K}} - \hat{\mathcal{K}}_c = \frac{1}{4}\left[\hat{\mathcal{K}}^B, \left[\ln\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right), \hat{\mathcal{K}}_r^B\right]\right]\,. \tag{51}$$

It means that indeed conformal and standard forms of the kernel are connected by a similarity transformation. Moreover, this transformation is equivalent to the change of the subtraction procedure in the definition of the NLO kernel [19].

5 Conclusion

In this paper, we have shown that the standard form of the modified BFKL kernel (i.e. the BFKL kernel in $N = 4$ SUSY for the adjoint representation of the gauge group with subtracted gluon trajectory depending on total momentum transfer) can be reduced by a similarity transformation to a form which is Möbius invariant in the momentum space. The transformation is given by Eq. (51). This transformation is equivalent to the change of the subtraction procedure for separation of leading and next-to-leading contributions used in the definition of the NLO kernel [19].

The Möbius invariant kernel was used for calculation of the NLO remainder function in [17] with the Möbius invariant convolution of the NLO BFKL impact factors (which was called for brevity simply impact factor) obtained in [14] from direct two-loop calculations and with the energy scale s_0 chosen in such a way that the ratio $s/s_0 = 1/\sqrt{u_2 u_3}$ is Möbius invariant. In principle, one can use different definitions of s_0 with Möbius invariant ratio s/s_0. The definition used in [17] is natural because of t-channel factorization of the amplitude in the Regge theory and is matched to the definition of the NLO BFKL kernel and impact factors [19]. But for complete assurance in the Möbius invariance of the remainder function, one should check that the convolution of the last impact factors is reduced to the impact factor used in [17] by the same similarity transformation as the kernel.

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