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ON THE DISCREPANCY OF THE LOW-X EVOLUTION KERNELS

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On the discrepancy of the low-x evolution kernels *

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Abstract

It is shown that in the case of the forward scattering the most part of the difference between the Möbius form of the BFKL kernel and the BK kernel in the next-to-leading order (NLO) can be eliminated by the transformation related to the choice of the energy scale in the representation of scattering amplitudes. Change of the nonforward BFKL kernel under this transformation is derived as well. The functional identity of the forward BFKL kernel in the momentum and Möbius representations in the leading order (LO) is exhibited and its NLO validity in $N = 4$ supersymmetric Yang-Mills theory is proved.

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1 Introduction

The Balitsky-Fadin-Kuraev-Lipatov (BFKL) framework for description of the semihard processes in QCD was developed originally in the momentum representation. In the leading order (LO) it was done in Refs. [1, 2]; the next to leading (NLO) corrections to the kernel of the BFKL equation were obtained in Refs. [3] - [7]. The field of applicability of the BFKL approach is quite wide. It includes scattering processes with arbitrary colour exchanges.

For scattering of colourless particles an alternative framework to the description of the semihard processes is the colour dipole model [8]. In contrast to the BFKL approach this model is formulated in the impact parameter space. The equation describing in the LO evolution of scattering amplitudes with energy in this model has a simple non-linear generalization, which is called Balitsky-Kovchegov (BK) equation [9]. The NLO corrections to the kernel of the BK equation in the coordinate space have been calculated recently in Refs. [10] -[12].

These two approaches should give the same predictions in the common area. This requirement is definitely fulfilled in the LO. Indeed, for scattering of colourless particles the LO BFKL kernel can be taken in the Möbius representation, which is invariant in regard to the conformal transformations of the transverse coordinates [13]. In the impact parameter space the Möbius representation of the BFKL kernel coincides with the kernel of the colour dipole model [14] (it was called therefore in [14] dipole form of the BFKL kernel). Moreover, it was shown [15] that the LO BK equation appears as a special case of the nonlinear evolution equation which sums the fun diagrams for the BFKL Green's functions in the Möbius representation.

In the NLO one could expect coincidence of the Möbius form of the BFKL kernel and the kernel of the linearized BK equation. However, the situation is not so simple. First of all, the NLO kernels are not unambiguously defined. The ambiguity of the NLO kernels is analogous to the ambiguity on the NLO anomalous dimensions. It is caused by the possibility to redistribute radiative corrections between the kernels and the impact factors. This freedom in the definition of the kernels allows one to reshape the Möbius form of the BFKL kernel in order to prove its equivalence to the BK one [14].

In QCD the NLO kernels consist of two parts: the quark and the gluon ones. The quark part of the BK kernel was found in Ref. [10] and [11]. The Möbius form of the quark part of the BFKL kernel was obtained in Refs. [14] amd [16] from the quark contribution to the BFKL kernel calculated in the momentum representation in Ref. [5]. It was proved [14, 16] that with account of the freedom mentioned above this form is equivalent to the quark part of the linearized BK kernel. The Möbius form of the gluon part was obtained in Ref. [17] with use of the gluon correction to the BFKL kernel calculated in Refs. [6] and [7] in the momentum representation. The gluon correction to the BK kernel was found in Ref. [12]. It occured that for the linearized BK equation this correction strongly differs from the Möbius form obtained in Ref. [17].

In this paper we demonstrate that for the case of forward scattering the most part of the difference can be eliminated by the transformation related to the choice of the energy scale in the representation of scattering amplitudes. We also show how the simplest generalizations of this transformation change the nonforward kernel.

The second goal of our work is to exhibit the functional identity of the forward BFKL kernels in the momentum and the Möbius representations and to prove that in the N=4 SUSY Yang-Mills theory this identity remains valid in the NLO. The extension of the BFKL framework to the supersymmetric theories was started in [18], where the kernel for the forward scattering was found for the N=4 SUSY in the momentum space with the dimension $D = 4+2\epsilon$ and in the eigenfunction space. This analysis has been recently expanded in Ref. [19], where the nonforward Möbius NLO BFKL kernel was obtained for the SUSY Yang-Mills theories with any N. We continue this line finding the forward kernel in the momentum space for any N, writing it at $D = 4$, with all singularities cancelled, and demonstrating the functional identity of this kernel to the forward kernel in the Möbius representation for N=4.

Our paper is organized as follows. The next section introduces our notation and gives a brief account of the main results of Ref. [17]. Section 3 goes through different transforms of the kernel which do not change observables. Section 4 shows that the difference between the Möbius form of the BFKL kernel and the kernel of the linearized BK equation can be partially eliminated by the suitable transform. Section 5 demonstrates the functional identity of the forward BFKL kernels in the momentum and Möbius coordinate representations. Section 6 discusses how the generalizations of the transformation obtained in section 4 change the nonforward kernel. Section 7 summarizes the main points of our paper. Appendices describe the details of the calculations.

2 General overview

Our notation is the same as in Refs. [14] and [17]. Thus the Reggeon transverse momenta (and the conjugate coordinates) in initial and final t-channel states are $\vec{q}_i^{\,\prime}$ $(\vec{r}_i^{\,\prime})$ and $\vec{q}_i$ $(\vec{r}_i)$, $i=1,2$. The state normalization is

$$\langle\vec{q}|\vec{q}\,'\rangle=\delta(\vec{q}-\vec{q}\,')\;,\qquad\langle\vec{r}|\vec{r}\,'\rangle=\delta(\vec{r}-\vec{r}\,')\;,\qquad\langle\vec{r}|\vec{q}\rangle=\frac{e^{i\vec{q}\,\vec{r}}}{(2\pi)^{1+\epsilon}}. \tag{1}$$

Here $\epsilon=(D-4)/2$; $D-2$ is the transverse space dimension taken different from 2 to regularize divergences. We will also write for brevity $\vec{p}_{ij'}=\vec{p}_i-\vec{p}_j^{\,\prime}$.

The s-channel discontinuities of scattering amplitudes for the processes $A+B\to A'+B'$ have the form

$$-4i(2\pi)^{D-2}\delta(\vec{q}_A-\vec{q}_B)\mathrm{disc}_s\mathcal{A}_{AB}^{A'B'}=\langle A'\bar{A}|\left(\frac{s}{s_0}\right)^{\hat{\mathcal{K}}}\frac{1}{\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}}|\bar{B}'B\rangle\;. \tag{2}$$

In this expression s_0 is an appropriate energy scale, $\hat{\mathcal{K}}$ is the BFKL kernel, $q_A=p_{A'A},\;\; q_B=p_{BB'}$, and

$$\begin{aligned}\langle\vec{q}_1,\vec{q}_2|\hat{\mathcal{K}}\,|\vec{q}_1^{\,\prime},\vec{q}_2^{\,\prime}\rangle &= \delta(\vec{q}_1+\vec{q}_2-\vec{q}_1^{\,\prime}-\vec{q}_2^{\,\prime})\frac{\mathcal{K}_r(\vec{q}_1,\vec{q}_1^{\,\prime};\vec{q})}{\vec{q}_1^{\,2}\vec{q}_2^{\,2}}\\ &\quad+\delta(\vec{q}_{22'})\delta\left(\vec{q}_{11'}\right)\left(\omega\left(\vec{q}_1^{\,2}\right)+\omega(\vec{q}_2^{\,2})\right),\end{aligned} \tag{3}$$

with $\omega(t)$ being the gluon Regge trajectory, and $\hat{\mathcal{K}}_r$ representing real particle production in Reggeon collisions,

$$\langle\vec{q}_1,\vec{q}_2|\bar{B}'B\rangle=4p_B^-\delta(\vec{q}_B-\vec{q}_1-\vec{q}_2)\Phi_{B'B}(\vec{q}_1,\vec{q}_2)\;, \tag{4}$$

$$\langle A'\bar{A}|\vec{q}_1,\vec{q}_2\rangle=4p_A^+\delta(\vec{q}_A-\vec{q}_1-\vec{q}_2)\Phi_{A'A}(\vec{q}_1,\vec{q}_2)\;, \tag{5}$$

where $p^\pm=(p_0\pm p_z)/\sqrt{2}$; the kernel $\mathcal{K}_r(\vec{q}_1,\vec{q}_1^{\,\prime};\vec{q})$ and the impact factors Φ are expressed through the Reggeon vertices according to Ref. [20].

The general form of the Möbius (dipole) kernel in the coordinate representation reads [14, 17]:

$$\langle\vec{r}_1\vec{r}_2|\hat{\mathcal{K}}_M|\vec{r}_1^{\,\prime}\vec{r}_2^{\,\prime}\rangle=\frac{\alpha_s(\mu^2)N_c}{2\pi^2}\int d\vec{\rho}\,\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{1\rho}^{\,2}\vec{r}_{2\rho}^{\,2}}$$

$$\times\left[\delta(\vec{r}_{11'})\delta(\vec{r}_{2'\rho})+\delta(\vec{r}_{1'\rho})\delta(\vec{r}_{22'})-\delta(\vec{r}_{11'})\delta(r_{22'})\right]$$

$$+\frac{\alpha_s^2(\mu^2)N_c^2}{4\pi^3}\Bigg[\delta(\vec{r}_{11'})\delta(\vec{r}_{22'})\int d\vec{\rho}\, g^0(\vec{r}_1,\vec{r}_2;\vec{\rho})$$

$$+\delta(\vec{r}_{11'})g(\vec{r}_1,\vec{r}_2;\vec{r}_2^{\,\prime})+\delta(\vec{r}_{22'})g(\vec{r}_2,\vec{r}_1;\vec{r}_1^{\,\prime})+\frac{1}{\pi}g(\vec{r}_1,\vec{r}_2;\vec{r}_1^{\,\prime},\vec{r}_2^{\,\prime})\Bigg]. \qquad (6)$$

Here $\vec{r}_{i\rho}=\vec{r}_i-\vec{\rho}$, and the whole kernel is symmetric with respect to simultaneous $1\leftrightarrow 2$ and $1'\leftrightarrow 2'$ substitution.

The quark contribution to the functions g was calculated in Refs. [14, 16] and after a suitable transform was shown to coincide with the BK result of Ref. [11]. The gluon contribution to these functions was found in the BFKL approach in Ref. [17] and in the dipole one in Ref. [12]. The latter two results are different.

The BFKL framework gives for the gluon contribution

$$g^0(\vec{r}_1,\vec{r}_2;\vec{\rho})=-g(\vec{r}_1,\vec{r}_2;\vec{\rho})\ +2\pi\zeta(3)\delta\left(\vec{\rho}\right), \qquad (7)$$

$$g(\vec{r}_1,\vec{r}_2;\vec{r}_2^{\,\prime})\ =\frac{11}{6}\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{22'}^{\,2}\vec{r}_{12'}^{\,2}}\ln\left(\frac{\vec{r}_{12}^{\,2}}{r_\mu^2}\right)+\frac{11}{6}\left(\frac{1}{\vec{r}_{22'}^{\,2}}-\frac{1}{\vec{r}_{12'}^{\,2}}\right)\ln\left(\frac{\vec{r}_{22'}^{\,2}}{\vec{r}_{12'}^{\,2}}\right)$$

$$+\frac{1}{2\vec{r}_{22'}^{\,2}}\ln\left(\frac{\vec{r}_{12'}^{\,2}}{\vec{r}_{22'}^{\,2}}\right)\ln\left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{12'}^{\,2}}\right)-\frac{\vec{r}_{12}^{\,2}}{2\,\vec{r}_{22'}^{\,2}\vec{r}_{12'}^{\,2}}\ln\left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{22'}^{\,2}}\right)\ln\left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{12'}^{\,2}}\right), \qquad (8)$$

where

$$\ln r_\mu^2=-2C-\ln\frac{\mu^2}{4}-\frac{3}{11}\left(\frac{67}{9}-2\zeta(2)\right) \qquad (9)$$

and $C=-\psi(1)$ is the Euler constant; μ is a renormalization scale in the $\overline{MS}$-scheme. At once we will emphasize that only the integral of $g^0(\vec{r}_1,\vec{r}_2;\rho)$ contributes to the kernel. Therefore one can write g^0 in various forms, e.g. in our previous papers we used the equalities

$$\int d\vec{\rho}\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{1\rho}^{\,2}\vec{r}_{2\rho}^{\,2}}\ln\left(\frac{\vec{r}_{1\rho}^{\,2}}{\vec{r}_{12}^{\,2}}\right)\ln\left(\frac{\vec{r}_{2\rho}^{\,2}}{\vec{r}_{12}^{\,2}}\right)=\int\frac{d\vec{\rho}}{\vec{r}_{2\rho}^{\,2}}\ln\left(\frac{\vec{r}_{1\rho}^{\,2}}{\vec{r}_{12}^{\,2}}\right)\ln\left(\frac{\vec{r}_{1\rho}^{\,2}}{\vec{r}_{2\rho}^{\,2}}\right)=4\pi\zeta(3) \qquad (10)$$

and

$$\int d\vec{\rho}\left[\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{1\rho}^{\,2}\vec{r}_{2\rho}^{\,2}}\ln\left(\frac{\vec{r}_{1\rho}^{\,2}\vec{r}_{2\rho}^{\,2}}{\vec{r}_{12}^{\,4}}\right)+\left(\frac{1}{\vec{r}_{2\rho}^{\,2}}-\frac{1}{\vec{r}_{1\rho}^{\,2}}\right)\ln\left(\frac{\vec{r}_{1\rho}^{\,2}}{\vec{r}_{2\rho}^{\,2}}\right)\right]=0, \qquad (11)$$

to reshape it. But anyway $g^0(\vec{r}_1,\vec{r}_2;\rho)$ does not coincide with $g(\vec{r}_1,\vec{r}_2;\rho)$. For the third function we have

$$g(\vec{r}_1,\vec{r}_2;\vec{r}_1^{\,\prime},\vec{r}_2^{\,\prime}) =$$

$$= \frac{1}{2\vec{r}^{\,4}_{1'2'}}\left(\frac{\vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{22'}-2\vec{r}^{\,2}_{12}\vec{r}^{\,2}_{1'2'}}{d}\ln\left(\frac{\vec{r}^{\,2}_{12'}\vec{r}^{\,2}_{21'}}{\vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{22'}}\right)-1\right)+\frac{\vec{r}^{\,2}_{12}\ln\left(\frac{\vec{r}^{\,2}_{11'}}{\vec{r}^{\,2}_{1'2'}}\right)}{2\vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{12'}\vec{r}^{\,2}_{22'}}$$
$$+\frac{\ln\left(\frac{\vec{r}^{\,2}_{12'}\vec{r}^{\,2}_{21'}}{\vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{22'}}\right)}{4\vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{22'}}\left(\frac{\vec{r}^{\,4}_{12}}{d}-\frac{\vec{r}^{\,2}_{12}}{\vec{r}^{\,2}_{1'2'}}\right)+\frac{\ln\left(\frac{\vec{r}^{\,2}_{12}\vec{r}^{\,2}_{1'2'}}{\vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{22'}}\right)}{2\vec{r}^{\,2}_{12'}\vec{r}^{\,2}_{21'}}\left(\frac{\vec{r}^{\,2}_{12}}{2\vec{r}^{\,2}_{1'2'}}+\frac{1}{2}-\frac{\vec{r}^{\,2}_{22'}}{\vec{r}^{\,2}_{1'2'}}\right)$$
$$+\frac{\vec{r}^{\,2}_{21'}\ln\left(\frac{\vec{r}^{\,2}_{21'}\vec{r}^{\,2}_{1'2'}}{\vec{r}^{\,2}_{12}\vec{r}^{\,2}_{11'}}\right)}{2\vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{22'}\vec{r}^{\,2}_{1'2'}}+\frac{\ln\left(\frac{\vec{r}^{\,2}_{12}}{\vec{r}^{\,2}_{1'2'}}\right)}{4\vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{22'}}+\frac{\ln\left(\frac{\vec{r}^{\,2}_{22'}}{\vec{r}^{\,2}_{12}}\right)}{2\vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{12'}}+\frac{\vec{r}^{\,2}_{12}\ln\left(\frac{\vec{r}^{\,2}_{12}\vec{r}^{\,2}_{1'2'}}{\vec{r}^{\,2}_{12'}\vec{r}^{\,2}_{21'}}\right)}{4\vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{22'}\vec{r}^{\,2}_{1'2'}}$$
$$+\frac{\ln\left(\frac{\vec{r}^{\,2}_{12}\vec{r}^{\,2}_{1'2'}}{\vec{r}^{\,2}_{12'}\vec{r}^{\,2}_{22'}}\right)}{2\vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{1'2'}}+\frac{\ln\left(\frac{\vec{r}^{\,2}_{12}\vec{r}^{\,2}_{11'}}{\vec{r}^{\,2}_{22'}\vec{r}^{\,2}_{1'2'}}\right)}{2\vec{r}^{\,2}_{12'}\vec{r}^{\,2}_{1'2'}}+(1\leftrightarrow 2, 1'\leftrightarrow 2'), \tag{12}$$

where

$$d=\vec{r}^{\,2}_{12'}\vec{r}^{\,2}_{21'}-\vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{22'}. \tag{13}$$

The functions $g(\vec{r}_1,\vec{r}_2;\vec{r}_2^{\,\prime})$ and $g(\vec{r}_1,\vec{r}_2;\vec{r}_1^{\,\prime},\vec{r}_2^{\,\prime})$ vanish at $\vec{r}_1=\vec{r}_2$. In Ref. [17] the latter term was presented in a form where this property was obvious. For integration, however, it is more convenient to use the expression (12) .

At the same time the gluon part of the kernel of the linearized BK equation gives [12] :

$$g^0_{BC}(\vec{r}_1,\vec{r}_2,\vec{\rho}) = -g_{BC}(\vec{r}_1,\vec{r}_2,\vec{\rho})\ , \tag{14}$$

$$g_{BC}(\vec{r}_1,\vec{r}_2;\vec{r}_2^{\,\prime}) = \frac{11}{6}\frac{\vec{r}^{\,2}_{12}}{\vec{r}^{\,2}_{22'}\vec{r}^{\,2}_{12'}}\ln\left(\frac{\vec{r}^{\,2}_{12}}{r^2_{\mu_B}}\right)+\frac{11}{6}\left(\frac{1}{\vec{r}^{\,2}_{22'}}-\frac{1}{\vec{r}^{\,2}_{12'}}\right)\ln\left(\frac{\vec{r}^{\,2}_{22'}}{\vec{r}^{\,2}_{12'}}\right)$$
$$-\frac{\vec{r}^{\,2}_{12}}{\vec{r}^{\,2}_{22'}\vec{r}^{\,2}_{12'}}\ln\left(\frac{\vec{r}^{\,2}_{12}}{\vec{r}^{\,2}_{22'}}\right)\ln\left(\frac{\vec{r}^{\,2}_{12}}{\vec{r}^{\,2}_{12'}}\right), \tag{15}$$

where

$$\ln r^2_{\mu_{BC}} = -\ln\mu^2-\frac{3}{11}\left(\frac{67}{9}-2\zeta(2)\right). \tag{16}$$

$$g_{BC}(\vec{r}_1,\vec{r}_2;\vec{r}_1^{\,\prime},\vec{r}_2^{\,\prime}) = \ln\left(\frac{\vec{r}^{\,2}_{12'}\vec{r}^{\,2}_{21'}}{\vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{22'}}\right)\left[\frac{\vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{22'}+\vec{r}^{\,2}_{12'}\vec{r}^{\,2}_{21'}-4\vec{r}^{\,2}_{12}\vec{r}^{\,2}_{1'2'}}{2d\,\vec{r}^{\,4}_{1'2'}}\right.$$
$$\left.+\frac{1}{4\vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{22'}}\left(\frac{\vec{r}^{\,4}_{12}}{d}-\frac{\vec{r}^{\,2}_{12}}{\vec{r}^{\,2}_{1'2'}}\right)+\frac{1}{4\vec{r}^{\,2}_{12'}\vec{r}^{\,2}_{21'}}\left(\frac{\vec{r}^{\,4}_{12}}{d}+\frac{\vec{r}^{\,2}_{12}}{\vec{r}^{\,2}_{1'2'}}\right)\right]-\frac{1}{\vec{r}^{\,4}_{1'2'}}. \tag{17}$$

3 Ambiguity in the definition of the kernel

To begin with, we are going to discuss the transformations of the kernel which do not affect observables.

First of all, $\mathrm{disc}_s\mathcal{A}_{AB}^{A'B'}$ in Eq. (2) remains intact if one changes both the kernel and the impact factors via an arbitrary nonsingular operator $\hat{\mathcal{O}}$:

$$\hat{\mathcal{K}} \to \hat{\mathcal{O}}^{-1}\hat{\mathcal{K}}\hat{\mathcal{O}} \ , \ \ \langle A'\bar{A}| \to \langle A'\bar{A}|\hat{\mathcal{O}} \ , \ \ \frac{1}{\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}}|\bar{B}'B\rangle \to \hat{\mathcal{O}}^{-1}\frac{1}{\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}}|\bar{B}'B\rangle. \quad (18)$$

If the LO kernel is fixed by the requirement that its Möbius form coincides with the kernel of the dipole model, one can shift the NLO contribution using $\hat{\mathcal{O}} = 1 - \hat{O}$, with $\hat{O} \sim \alpha_s$, and get

$$\hat{\mathcal{K}} \to \hat{\mathcal{K}} - [\hat{\mathcal{K}}^{(B)}, \hat{O}], \quad (19)$$

where the superscript (B) means the LO kernel. Note that the Möbius form calculated in Refs. [14, 17] and presented in the previous section corresponds to the kernel obtained by the transformation

$$\hat{\mathcal{K}} \to \hat{\mathcal{K}} + \frac{\alpha_s}{8\pi}\beta_0[\hat{\mathcal{K}}^{(B)}, \ln\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right)] \ , \quad (20)$$

where β_0 is the first coefficient of the beta-function, from the kernel defined in (3). This transformation simplifies the Möbius form and allows to alter the quark part of this form so that it agrees with the result of Ref. [11].

Secondly, there is a freedom in the energy scale s_0. At first sight, it can lead to an additional ambiguity of the NLO kernel. However, it is not so. Indeed, it was shown [21] that any change of the energy scale can be compensated by corresponding redefinition of the impact factors. An experienced reader can wonder remembering that in Ref. [3] the scale transformation was associated with the change of the kernel. The matter is that in Ref. [3] one of the impact factors was fixed. Actually, instead of transformation of both impact factors one can compensate any change of the scale by transformation of one of the impact factors and the kernel. Evidently, in this case the transformation of the kernel has the form (19) with some specific form of the operator $\hat{O}$. Let us discuss this question more in detail.

We begin with the case when s_0 depends only on properties of scattering particles. Just this case was supposed at the definition of the kernel and the impact factors in Ref. [20]. Note that s_0 can be taken as a free parameter. This freedom can be used for optimization of perturbative results [22]. A natural choice is $s_0 = Q_AQ_B$, where Q_A and Q_B are typical virtualities for the impact factors $\Phi_{A'A}$ and $\Phi_{B'B}$ correspondingly. Let us consider the transition from such scale to the scale depending on the Reggeon momenta $\vec{q}_{Ai}$ and $\vec{q}_{Bi}$, $i = 1, 2$ in the impact factors $\Phi_{A'A}$ and $\Phi_{B'B}$ respectively:

$$s_0 \to f_A f_B, \quad f_A \equiv f_A(\vec{q}_{Ai}), \quad f_B \equiv f_B(\vec{q}_{Bi}). \tag{21}$$

Remind that with the NLO accuracy for any s-independent value c one has

$$c^{\hat{\mathcal{K}}} = 1 + \hat{\mathcal{K}}^{(B)} \ln c \; . \tag{22}$$

Therefore one can write

$$\langle \vec{q}_{A1}, \vec{q}_{A2} | \left(\frac{s}{s_0} \right)^{\hat{\mathcal{K}}} | \vec{q}_{B1}, \vec{q}_{B2} \rangle = \langle \vec{q}_{A1}, \vec{q}_{A2} | \hat{F}_A \left(\frac{s}{f_A f_B} \right)^{\hat{\mathcal{K}}} \hat{F}_B | \vec{q}_{B1}, \vec{q}_{B2} \rangle, \tag{23}$$

where

$$\hat{F}_A = \left(1 + \ln \left(\frac{\hat{f}_A}{s_0^{\alpha}} \right) \hat{\mathcal{K}}^{(B)} \right), \quad \hat{F}_B = \left(1 + \hat{\mathcal{K}}^{(B)} \ln \left(\frac{\hat{f}_A}{s_0^{\beta}} \right) \right),$$

$$\alpha + \beta = 1, \quad \hat{f}_A \equiv f_A(\hat{\vec{q}}_i), \quad \hat{f}_B \equiv f_B(\hat{\vec{q}}_i). \tag{24}$$

It means that the discontinuity (2) remains unchanged if the change of the scale (23) is accompanied by the change of the impact factors

$$\langle A'\bar{A}| \to \langle A'\bar{A}|\hat{F}_A, \quad \frac{1}{\hat{\vec{q}}_1^{\,2} \hat{\vec{q}}_2^{\,2}} |\bar{B}'B\rangle \to \hat{F}_B \frac{1}{\hat{\vec{q}}_1^{\,2} \hat{\vec{q}}_2^{\,2}} |\bar{B}'B\rangle. \tag{25}$$

It is possible to leave one of the impact factors (let us take for definiteness $\langle A'\bar{A}|$) invariable changing the kernel. Indeed,

$$\hat{F}_A \left(\frac{s}{f_A f_B} \right)^{\hat{\mathcal{K}}} \hat{F}_A^{-1} = \left(\frac{s}{f_A f_B} \right)^{\hat{\mathcal{K}}'}, \quad \hat{\mathcal{K}}' = \hat{F}_A \hat{\mathcal{K}} \hat{F}_A^{-1} \; . \tag{26}$$

Therefore instead of the change (25) one can take

$$\hat{\mathcal{K}} \to \hat{\mathcal{K}}', \quad \frac{1}{\hat{\vec{q}}_1^{\,2} \hat{\vec{q}}_2^{\,2}} |\bar{B}'B\rangle \to \hat{F}_A \hat{F}_B \frac{1}{\hat{\vec{q}}_1^{\,2} \hat{\vec{q}}_2^{\,2}} |\bar{B}'B\rangle, \tag{27}$$

where, with the NLO accuracy,

$$\hat{\mathcal{K}}' = \hat{\mathcal{K}} - \left[\hat{\mathcal{K}}^{(B)}, \ln \hat{f}_A \hat{\mathcal{K}}^{(B)} \right] \; . \tag{28}$$

We see that the change of the energy scale can be associated with the transformation of the kernel (19) with the specific $\hat{O}$.

At last, the Möbius kernel (6) is defined with an accuracy to any function independent of $\vec{r}_1$ or of $\vec{r}_2$ such that after their addition the kernel remains zero at $\vec{r}_1 = \vec{r}_2$ [14, 17]. Therefore, one can add to the kernel only the functions which are antisymmetric with respect to the substitution $\vec{r}_1{}' \leftrightarrow \vec{r}_2{}'$. These functions do not change the symmetric part of the kernel. But this part alone plays a role in the observables. As a result, the third term in expression (12) in the BFKL kernel,

$$\frac{\ln\left(\frac{\vec{r}^{\,2}_{12'}\,\vec{r}^{\,2}_{21'}}{\vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{22'}}\right)}{4\vec{r}^{\,2}_{11'}\,\vec{r}^{\,2}_{22'}}\left(\frac{\vec{r}^{\,4}_{12}}{d}-\frac{\vec{r}^{\,2}_{12}}{\vec{r}^{\,2}_{1'2'}}\right)+(1\leftrightarrow 2, 1'\leftrightarrow 2') \tag{29}$$

and the term

$$\ln\left(\frac{\vec{r}^{\,2}_{12'}\vec{r}^{\,2}_{21'}}{\vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{22'}}\right)\left[\frac{1}{4\vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{22'}}\left(\frac{\vec{r}^{\,4}_{12}}{d}-\frac{\vec{r}^{\,2}_{12}}{\vec{r}^{\,2}_{1'2'}}\right)+\frac{1}{4\vec{r}^{\,2}_{12'}\vec{r}^{\,2}_{21'}}\left(\frac{\vec{r}^{\,4}_{12}}{d}+\frac{\vec{r}^{\,2}_{12}}{\vec{r}^{\,2}_{1'2'}}\right)\right] \tag{30}$$

in the BK kernel (17) give the same contribution to the amplitudes.

The ambiguities of the NLO kernels give a hope that the results of Refs. [17] and [12] can be matched.

4 The kernel for forward scattering in gluodynamics

For a start let us find the gluon contribution to the forward Möbius BFKL kernel and compare it to the BK result obtained in [12]. We define the matrix element of the forward kernel in the momentum representation as

$$\langle\vec{q}|\hat{\mathcal{K}}|\vec{q}\,'\rangle = \int \langle\vec{q},\vec{l}|\hat{\mathcal{K}}\,|\vec{q}\,', -\vec{q}\,'\rangle d\vec{l}. \tag{31}$$

Next, using the state normalization (1) and the denotation $\vec{r}_1 = \vec{r}+\vec{r}_2$, $\vec{r}_1{}' = \vec{r}\,' + \vec{r}_2{}'$, we get at physical value $D = 4$

$$\langle\vec{q}|\hat{\mathcal{K}}|\vec{q}\,'\rangle = \int \frac{d\vec{r}}{2\pi}\frac{d\vec{r}\,'}{2\pi} e^{-i\vec{q}\vec{r}+i\vec{q}\,'\vec{r}\,'}\langle\vec{r}|\hat{\mathcal{K}}|\vec{r}\,'\rangle \ , \tag{32}$$

where

$$\langle\vec{r}|\hat{\mathcal{K}}|\vec{r}\,'\rangle = \int d\vec{r}_2{}'\langle\vec{r},\vec{0}|\hat{\mathcal{K}}|\vec{r}\,'+\vec{r}_2{}',\vec{r}_2{}'\rangle = \int \langle\vec{r}_1\vec{r}_2|\hat{\mathcal{K}}|\vec{r}_1{}'\vec{r}_2{}'\rangle\delta(\vec{r}_{1'2'}-\vec{r}\,')d\vec{r}_1{}'d\vec{r}_2{}' \ . \tag{33}$$

The last equality follows from the space uniformity.

Thus, the Möbius form of the BFKL kernel for the forward scattering can be obtained from Eq. (33) with use of the results of Refs. [14], [16] and [17] for $\langle \vec{r}_1\vec{r}_2|\hat{\mathcal{K}}_M|\vec{r}_1^{\,\prime}\vec{r}_2^{\,\prime}\rangle$. In the case of pure gluodynamics it gives

$$\langle\vec{r}|\hat{\mathcal{K}}_M|\vec{r}^{\,\prime}\rangle = \frac{\alpha_s(\frac{4e^{-2C}}{\vec{r}^{\,2}})N_c}{2\pi^2}\int\frac{d\vec{\rho}\,\vec{r}^{\,2}}{(\vec{r}-\vec{\rho})^2\vec{\rho}^{\,2}}\Bigg\{\Big(2\delta(\vec{\rho}-\vec{r}^{\,\prime})-\delta(\vec{r}-\vec{r}^{\,\prime})\Big)$$

$$\times\left[1+\frac{\alpha_sN_c}{4\pi}\left(\frac{67}{9}-2\zeta(2)+\frac{11}{3}\frac{\vec{\rho}^{\,2}-(\vec{r}-\vec{\rho})^2}{\vec{r}^{\,2}}\ln\left(\frac{(\vec{r}-\vec{\rho})^2}{\vec{\rho}^{\,2}}\right)\right)\right]$$

$$+\frac{\alpha_sN_c}{4\pi}3\delta(\vec{r}-\vec{r}^{\,\prime})\ln\left(\frac{(\vec{r}-\vec{\rho})^2}{\vec{r}^{\,2}}\right)\ln\left(\frac{\vec{\rho}^{\,2}}{\vec{r}^{\,2}}\right)\Bigg\}$$

$$+\frac{\alpha_s^2N_c^2}{4\pi^3}\frac{\vec{r}^{\,2}}{\vec{r}^{\,\prime 2}}\left(f_1(\vec{r},\vec{r}^{\,\prime})+f_2(\vec{r},\vec{r}^{\,\prime})-\frac{1}{(\vec{r}-\vec{r}^{\,\prime})^2}\ln^2\left(\frac{\vec{r}^{\,2}}{\vec{r}^{\,\prime 2}}\right)\right)\;. \qquad (34)$$

Here

$$\alpha_s(\frac{4e^{-2C}}{\vec{r}^{\,2}})\simeq\alpha_s(\mu^2)\left(1-\frac{\alpha_s(\mu^2)}{4\pi}\frac{11}{3}N_c\ln\left(\frac{4e^{-2C}}{\vec{r}^{\,2}\mu^2}\right)\right)\;, \qquad (35)$$

μ is the renormalization scale in the $\overline{MS}$-scheme,

$$f_1(\vec{x},\vec{y}) = \frac{(\vec{x}^{\,2}-\vec{y}^{\,2})}{(\vec{x}-\vec{y})^2(\vec{x}+\vec{y})^2}\left[\ln\left(\frac{\vec{x}^{\,2}}{\vec{y}^{\,2}}\right)\ln\left(\frac{\vec{x}^{\,2}\vec{y}^{\,2}(\vec{x}-\vec{y})^4}{(\vec{x}^{\,2}+\vec{y}^{\,2})^4}\right)\right.$$

$$\left.+2\,\mathrm{Li}_2\left(-\frac{\vec{y}^{\,2}}{\vec{x}^{\,2}}\right)-2\,\mathrm{Li}_2\left(-\frac{\vec{x}^{\,2}}{\vec{y}^{\,2}}\right)\right]$$

$$-\left(1-\frac{(\vec{x}^{\,2}-\vec{y}^{\,2})^2}{(\vec{x}-\vec{y})^2(\vec{x}+\vec{y})^2}\right)\left[\int_0^1-\int_1^\infty\right]\frac{du}{(\vec{x}-\vec{y}u)^2}\ln\left(\frac{u^2\vec{y}^{\,2}}{\vec{x}^{\,2}}\right)\;, \qquad (36)$$

$$f_2(\vec{x},\vec{y}) = \frac{1}{8\vec{x}^{\,2}\vec{y}^{\,2}}\Bigg\{(\vec{x}\,\vec{y})^2\left(1-\frac{3}{2}\left(\frac{\vec{y}^{\,2}}{\vec{x}^{\,2}}+\frac{\vec{x}^{\,2}}{\vec{y}^{\,2}}\right)\right)$$

$$+(\vec{x}^{\,2}+\vec{y}^{\,2})^2-32\vec{x}^{\,2}\vec{y}^{\,2}\Bigg\}\int_0^\infty dt\frac{\ln\left|\frac{1+t}{1-t}\right|}{\vec{y}^{\,2}+t^2\vec{x}^{\,2}}$$

$$+\frac{3(\vec{x}\,\vec{y})^2-2\vec{x}^{\,2}\vec{y}^{\,2}}{16\vec{x}^{\,2}\vec{y}^{\,2}}\left(\ln\frac{\vec{x}^{\,2}}{\vec{y}^{\,2}}\left(\frac{1}{\vec{y}^{\,2}}-\frac{1}{\vec{x}^{\,2}}\right)+\frac{2}{\vec{x}^{\,2}}+\frac{2}{\vec{y}^{\,2}}\right)\;. \qquad (37)$$

The details of the derivation are given in the appendix A.

The result of Ref. [12] for the forward case is:

$$\langle\vec r|\hat{\mathcal{K}}_{BC}|\vec{r}\,'\rangle =$$

$$= \frac{\alpha_s(\frac{1}{\vec r^{\,2}})N_c}{2\pi^2}\int\frac{d\vec\rho\,\vec r^{\,2}}{(\vec r-\vec\rho)^2\vec\rho^{\,2}}\Big(2\delta(\vec\rho-\vec r\,')-\delta(\vec r-\vec r\,')\Big)\Bigg[1+\frac{\alpha_sN_c}{4\pi}\Bigg(\frac{67}{9}-2\zeta(2)$$
$$+\frac{11}{3}\frac{\vec\rho^{\,2}-(\vec r-\vec\rho)^2}{\vec r^{\,2}}\ln\left(\frac{(\vec r-\vec\rho)^2}{\vec\rho^{\,2}}\right)-2\ln\left(\frac{(\vec r-\vec\rho)^2}{\vec r^{\,2}}\right)\ln\left(\frac{\vec\rho^{\,2}}{\vec r^{\,2}}\right)\Bigg)\Bigg]$$
$$+\frac{\alpha_s^2N_c^2}{4\pi^3}\frac{\vec r^{\,2}}{\vec r^{\,\prime 2}}\left(f_1(\vec r,\vec r\,')+f_2(\vec r,\vec r\,')\right)\ . \tag{38}$$

It was derived using the relation (33). Note that in the derivation the term

$$\frac{\alpha_s^2N_c^2}{4\pi^3}\frac{\vec r^{\,2}}{\vec r^{\,\prime 2}}\left(f_1(\vec r,\vec r\,')-f_1(\vec r,-\vec r\,')\right)$$

was omitted. As we discussed at the end of the previous section, it can be done since this term vanishes at $\vec r\,'\leftrightarrow-\vec r\,'$ (i.e. $\vec r_1\,'\leftrightarrow\vec r_2\,'$).

Thus, for the difference of the forward kernels one has

$$\langle\vec r|\hat{\mathcal{K}}_M-\hat{\mathcal{K}}_{BC}|\vec r\,'\rangle=\frac{\alpha_s^2N_c^2}{4\pi^3}\Bigg[\frac{\vec r^{\,2}}{(\vec r-\vec r\,')^2\vec r^{\,\prime 2}}\ln\left(\frac{\vec r^{\,2}}{\vec r^{\,\prime 2}}\right)\ln\left(\frac{\vec r^{\,2}\vec r^{\,\prime 2}}{(\vec r-\vec r\,')^4}\right)$$
$$+\delta(\vec r-\vec r\,')2\pi\zeta(3)\Bigg]+\frac{\alpha_sN_c}{4\pi}\frac{11}{3}\left(C-\ln 2\right)\langle\vec r|\hat{\mathcal{K}}_M^{(B)}|\vec r\,'\rangle\ , \tag{39}$$

where

$$\langle\vec r|\hat{\mathcal{K}}_M^{(B)}|\vec r\,'\rangle=\frac{\alpha_sN_c}{2\pi^2}\int\frac{d\vec\rho\,\vec r^{\,2}}{(\vec r-\vec\rho)^2\vec\rho^{\,2}}\Big(2\delta(\vec\rho-\vec r\,')-\delta(\vec r-\vec r\,')\Big)\ . \tag{40}$$

The term proportional to 11/3 is related to renormalization (remind that in pure gluodynamics $\beta_0=11N_c/3$). In our opinion, this term arose because the renormalization scheme used in Ref. [12] is not equivalent to conventional $\overline{MS}$-scheme.

It occurs that the term with logarithms can be eliminated by the transformation

$$\hat{\mathcal{K}}\to\hat{\mathcal{K}}+\frac12\left[\hat{\mathcal{K}}^{(B)},\,\ln(\hat{\vec q}^{\,2})\,\hat{\mathcal{K}}^{(B)}\right] \tag{41}$$

applied to the forward BFKL kernel. Indeed, the direct calculation given in the appendix B shows that

$$\langle\vec r|\left[\hat{\mathcal{K}}^{(B)},\,\ln(\hat{\vec q}^{\,2})\,\hat{\mathcal{K}}^{(B)}\right]_M|\vec r\,'\rangle=-\frac{\alpha_s^2N_c^2}{2\pi^3}\frac{\vec r^{\,2}}{(\vec r-\vec r\,')^2\vec r^{\,\prime 2}}\ln\left(\frac{\vec r^{\,2}}{\vec r^{\,\prime 2}}\right)\ln\left(\frac{\vec r^{\,2}\vec r^{\,\prime 2}}{(\vec r-\vec r\,')^4}\right). \tag{42}$$

Here the subscript M means the Möbius representation (i.e. vanishing of the matrix element at $\vec{r}=0$). Comparing with Eq. (28) we see that the transformation (41) corresponds to the change of the energy scale at fixed value of one of the impact factors. Actually, the transformation (41) is of the same type as in Ref. [3]:

$$\mathcal{K}(\vec{q},\vec{q}\,') \to \mathcal{K}(\vec{q},\vec{q}\,') + \frac{1}{2}\int d\vec{p}\ \mathcal{K}^{(B)}(\vec{q},\vec{p})\ \ln\frac{\vec{p}^{2}}{\vec{q}^{\,2}}\mathcal{K}^{(B)}(\vec{p},\vec{q}\,'). \qquad (43)$$

One can come to the transformation (41) from another side. The difference $\hat{\mathcal{K}}_M - \hat{\mathcal{K}}_{BC}$ has the same eigenfunctions

$$\langle\vec{r}\,'|n,\gamma\rangle \sim e^{in\phi_{\vec{r}\,'}}\left(\vec{r}\,'^{2}\right)^{\gamma}\ , \qquad (44)$$

where ϕ is the azimuthal angle, as the LO dipole kernel. The eigenvalues of the LO dipole kernel coincide with the eigenvalues of the LO BFKL kernel obtained in [2]

$$\omega_B(n,\gamma) = \frac{\alpha_s N_c}{\pi}\chi(n,\gamma)\ ,\quad \chi(n,\gamma) = 2\psi(1)-\psi(\gamma+\frac{n}{2})-\psi(1-\gamma+\frac{n}{2})\ . \qquad (45)$$

It becomes evident if we write the FBKL kernel for the forward scattering as

$$\langle\vec{q}|\hat{\mathcal{K}}^{(B)}|\vec{q\,'}\rangle = \frac{\alpha_s N_c}{2\pi^2}\left[\frac{2\vec{q}\,'^{2}}{(\vec{q}-\vec{q}\,')^{2}\,\vec{q}^{\,2}} - \delta(\vec{q}-\vec{q}\,')\int\frac{d\vec{l}\,\vec{q}^{\,2}}{(\vec{q}-\vec{l})^{2}\vec{l}^{\,2}}\right] \qquad (46)$$

and compare it with the dipole kernel

$$\langle\vec{r}|\hat{\mathcal{K}}_d|\vec{r\,'}\rangle = \frac{\alpha_s N_c}{2\pi^2}\left[\frac{2\vec{r}^{\,2}}{(\vec{r}-\vec{r}\,')^{2}\,\vec{r}\,'^{2}} - \delta(\vec{r}-\vec{r}\,')\int\frac{d\vec{\rho}\,\vec{r}^{\,2}}{(\vec{r}-\vec{\rho})^{2}\vec{\rho}^{\,2}}\right]\ . \qquad (47)$$

It is worthwhile to note here that the kernel (46) differs from usually used symmetric kernel; the former is obtained from the latter by the transformation $\hat{\mathcal{K}} \to \hat{\vec{q}}^{\,-2}\hat{\mathcal{K}}\hat{\vec{q}}^{\,2}$. For the non-forward case corresponding transformation is $\hat{\mathcal{K}} \to (\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2})^{-1/2}\hat{\mathcal{K}}(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2})^{1/2}$. Let us stress that just the transformed kernel can be written in the Möbius form $\langle\vec{r}_1\vec{r}_2|\hat{\mathcal{K}}_M^{(B)}|\vec{r}_1{}'\vec{r}_2{}'\rangle$, which is invariant in regard to the conformal transformations of the transverse coordinates [13] and coincides with the kernel of the colour dipole model $\langle\vec{r}_1\vec{r}_2|\hat{\mathcal{K}}_d|\vec{r}_1{}'\vec{r}_2{}'\rangle$ [14]. Moreover, because of this coincidence, in the forward case one can see from (46) and (47) the functional identity of the LO BFKL kernel in the momentum and Möbius coordinate representations: $\vec{q}^{\,2}\langle\vec{q}|\hat{\mathcal{K}}^{(B)}|\vec{q\,'}\rangle/\vec{q}\,'^{2}$ is represented by the same function as $\vec{r}\,'^{2}\langle\vec{r}|\hat{\mathcal{K}}_M^{(B)}|\vec{r\,'}\rangle/\vec{r}^{\,2}$.

The eigenvalues $\omega_B(n,\gamma)$ are associated usually with the eigenfunctions $e^{in\phi_{\vec{q}\,'}}\left(\vec{q}\,'^2\right)^{\gamma-2}$ in the momentum space, i.e. $e^{in\phi_{\vec{r}\,'}}\left(\vec{r}\,'^2\right)^{1-\gamma}$ in the coordinate space, so that the eigenvalues for (44) should be $\omega_B(n,1-\gamma)$. On the other hand, from the functional identity of (46) and (47) it is clear that the eigenvalues must be the same as for $e^{in\phi_{\vec{r}\,'}}\left(\vec{q}\,'^2\right)^{\gamma-2}$, i.e. $\omega_B(n,\gamma)$. Both requirements are fulfilled because $\omega_B(n,\gamma)=\omega_B(n,1-\gamma)$.

Using the integral (the calculation is discussed in the Appendix C)

$$\int\frac{d\vec{r}\,'}{2\pi}e^{in(\phi_{\vec{r}\,'}-\phi_{\vec{r}})}\left(\frac{\vec{r}\,'^2}{\vec{r}^{\,2}}\right)^{\gamma}\frac{\vec{r}^{\,2}}{(\vec{r}-\vec{r}\,')^2\vec{r}\,'^2}\ln\left(\frac{\vec{r}^{\,2}}{\vec{r}\,'^2}\right)\ln\left(\frac{\vec{r}^{\,2}\vec{r}\,'^2}{(\vec{r}-\vec{r}\,')^4}\right)=$$

$$=2\chi'\left(n,\gamma\right)\chi\left(n,\gamma\right)\ , \tag{48}$$

where χ' means derivative over γ, we obtain from (39)

$$\omega_M\left(n,\gamma\right)-\omega_{BC}\left(n,\gamma\right)=$$

$$=\frac{\alpha_s^2(\mu^2)N_c^2}{2\pi^2}\left[\chi'\left(n,\gamma\right)\chi\left(n,\gamma\right)+\frac{11}{3}\left(C-\ln 2\right)\chi(n,\gamma)+\zeta(3)\right]\ , \tag{49}$$

where $\omega_M\left(n,\gamma\right)-\omega_{BC}\left(n,\gamma\right)$ is the eigenvalue of the difference $\hat{\mathcal{K}}_M-\hat{\mathcal{K}}_{BC}$ corresponding the the eigenfunction $e^{in\phi_{\vec{r}\,'}}\left(\vec{r}\,'^2\right)^{\gamma}$. The first term in (49) can be written as

$$\frac{1}{2}\omega_B'\left(n,\gamma\right)\omega_B\left(n,\gamma\right)=-\frac{1}{2}[\omega_B,\frac{\partial}{\partial\gamma}\ \omega_B]. \tag{50}$$

In the space of the eigenfunctions $e^{in\phi_{\vec{r}\,'}}\left(\vec{r}\,'^2\right)^{\gamma}$ we have $\hat{\mathcal{K}}^{(B)}=\omega_B\left(n,\gamma\right)$ and $\ln(\hat{\vec{q}}^{\,2})=-\partial/\partial\gamma$, so that we obtain for the forward scattering

$$\hat{\mathcal{K}}_M-\hat{\mathcal{K}}_{BC}=\frac{1}{2}[\hat{\mathcal{K}}^{(B)},\ln(\hat{\vec{q}}^{\,2})\hat{\mathcal{K}}^{(B)}]+\hat{\mathcal{K}}^{(B)}\frac{11}{3}\frac{\alpha_s(\mu^2)N_c}{2\pi}\left(C-\ln 2\right)+\frac{\alpha_s^2(\mu^2)N_c^2}{2\pi^2}\zeta(3). \tag{51}$$

Evidently, the first term in (51) is eliminated by the transformation (41). The second one, as it was already pointed out, in our opinion is related to the difference of the renormalization scheme used in Ref. [12] with conventional $\overline{MS}$-scheme and can be eliminated by change of the scheme. [1] Unfortunately, we can not find the transformation suitable to eliminate the third term. We have to add that in the BFKL approach the term with $\zeta(3)$ passed through a

[1]We have to note that in fact this term is present in the difference between the eigenvalues of the NLO BFKL kernel and the linearized forward kernel of Ref. [12]. In the calculation of this difference presented in Ref. [12] this term is erroneously omitted at the transition from Eq. (120) to Eq. (122).

great number of verifications. In particular, this term is necessary for fulfillment of the bootstrap relations. Besides, it is confirmed by the calculation of the three-loop anomalous dimensions in Refs. [23, 24].

For completeness we present here the characteristic function $\omega_M(n,\gamma)$ of the kernel (34), defined by the relation

$$\int d\vec{r}\,'\langle\vec{r}|\hat{\mathcal{K}}_M|\vec{r}\,'\rangle e^{in\phi_{\vec{r}\,'}}\left(\vec{r}\,'^2\right)^\gamma = \omega_M(n,\gamma)e^{in\phi_{\vec{r}}}\left(\vec{r}\,^2\right)^\gamma \ . \tag{52}$$

Actually because of running coupling the functions $e^{in\phi_{\vec{r}\,'}}\left(\vec{r}\,'^2\right)^\gamma$ are not eigenfunctions of $\hat{\mathcal{K}}_M$ anymore, and $\omega_M(n,\gamma)$ contains $\ln\vec{r}\,^2$. In (52) it can be replaced by $\partial/\partial\gamma$. As the result we have

$$\omega_M(n,\gamma) = \frac{\alpha_s(\mu^2)N_c}{\pi}\chi(n,\gamma) + \frac{\alpha_s^2(\mu^2)N_c^2}{4\pi^2}\Bigg[6\zeta\left(3\right) - 2\Phi\left(n,\gamma\right) - 2\Phi\left(n,1-\gamma\right)$$

$$+F\left(n,\gamma\right) - \chi''\left(n,\gamma\right) + \left(\frac{67}{9} - 2\zeta(2)\right)\chi(n,\gamma) \tag{53}$$

$$+\frac{11}{3}\left(\chi(n,\gamma)\left(\frac{\partial}{\partial\gamma} - \ln\left(\frac{4e^{-2C}}{\mu^2}\right) - \frac{2\gamma}{\gamma^2 - \frac{n^2}{4}}\right) + \frac{\chi^2(n,\gamma)}{2} - \frac{\chi'(n,\gamma)}{2}\right)\Bigg],$$

where

$$\Phi\left(n,\gamma\right) = \int\limits_0^1 \frac{dt}{1+t}t^{\gamma-1+n/2}\left\{\frac{\pi^2}{12} - \frac{1}{2}\psi'\left(\frac{n+1}{2}\right) - \mathrm{Li}_2\left(t\right) - \mathrm{Li}_2\left(-t\right)\right. \tag{54}$$

$$\left.-\left(\psi\left(n+1\right) + C + \ln\left(1+t\right) + \sum_{k=1}^{\infty}\frac{\left(-t\right)^k}{k+n}\right)\ln t - \sum_{k=1}^{\infty}\frac{t^k}{\left(k+n\right)^2}\left[1-\left(-1\right)^k\right]\right\}$$

and

$$F(n,\gamma) = \frac{\pi^2\cos\left(\pi\gamma\right)}{\sin^2\left(\pi\gamma\right)\left(1-2\gamma\right)}\left(\frac{\gamma\left(1-\gamma\right)\left(\delta_{n,2}+\delta_{n,-2}\right)}{2\left(3-2\gamma\right)\left(1+2\gamma\right)}\right.$$

$$\left.-\left\{\frac{3\gamma\left(1-\gamma\right)+2}{\left(3-2\gamma\right)\left(1+2\gamma\right)} + 3\right\}\delta_{n,0}\right). \tag{55}$$

The calculation of the integral (52) is discussed in the Appendix C.

As it should be, the function $\omega_M(n,\gamma)$ differs from the corresponding function $\omega(n,1-\gamma)$ found in Ref. [18] for the BFKL kernel in the momentum representation only by the terms related with renormalization (i.e. proportional $\beta_0/N_c = 11/3$ for pure gluodynamics). Indeed, if the coupling was not running, $\omega_M(n,\gamma)$ and $\omega_M(n,1-\gamma)$ would be genuine eigenvalues corresponding to the same eigenstate and should coincide. Since $\omega_M(n,\gamma)$ was found using the results of Refs. [14, 16] and [17] and the calculation of $\omega(n,\gamma)$ in Ref. [18] is based on the results of Ref. [3], the relationship of $\omega_M(n,\gamma)$ and $\omega(n,\gamma)$ is the cross-check of all these results. The coincidence of the functions at $\beta_0 = 0$ gives a forcible argument in favour of their correctness. Moreover, the terms proportional to $\beta_0/N_c = 11/3$ in $\omega_M(n,\gamma)$ can be derived from the corresponding term in $\omega(n,\gamma)$. Let the state $|n,\gamma\rangle$ be defined by the equality $\langle\vec{r}|n,\gamma\rangle = e^{in\phi_{\vec{r}}}\left(\vec{r}^{\,2}\right)^{\gamma}$; then, using

$$\int_0^{2\pi} d\phi e^{in\phi+ia\cos\phi} = e^{in\pi/2}2\pi J_n(a)\ , \quad \int_0^{\infty} dx x^{\alpha} J_n(bx) = 2^{\alpha} b^{-\alpha-1}\frac{\Gamma(\frac{n+1+\alpha}{2})}{\Gamma(\frac{n+1-\alpha}{2})}\ , \tag{56}$$

where J_n is the n-th Bessel function, we have

$$\langle\vec{q}|n,\gamma\rangle = e^{in\phi_{\vec{q}}}\left(\vec{q}^{\,2}\right)^{-1-\gamma} e^{i\pi n/2}\frac{2^{2\gamma+1}\Gamma(\frac{n}{2}+1+\gamma)}{\Gamma(\frac{n}{2}-\gamma)}\ . \tag{57}$$

The β-dependent terms in $\omega(n,1-\gamma)$ are written as $\alpha_s^2 N_c \beta_0 R/(4\pi^2)$, where

$$R = -\ln\left(\frac{\vec{q}^{\,2}}{\mu^2}\right)\chi(n,\gamma) - \frac{\chi^2(n,\gamma)}{2} + \frac{\chi'(n,\gamma)}{2}\ . \tag{58}$$

Since

$$-\ln\left(\frac{\vec{q}^{\,2}}{\mu^2}\right)\chi(n,\gamma)\langle\vec{q}|n,\gamma\rangle$$

$$= \chi(n,\gamma)\left(\ln\mu^2 + \frac{\partial}{\partial\gamma} - \left[\frac{\partial}{\partial\gamma}\ln\left(\frac{2^{2\gamma+1}\Gamma(\frac{n}{2}+1+\gamma)}{\Gamma(\frac{n}{2}-\gamma)}\right)\right]\right)\langle\vec{q}|n,\gamma\rangle$$

$$= \chi(n,\gamma)\left(\ln\mu^2 + \frac{\partial}{\partial\gamma} - 2\ln 2 + 2C + \chi(n,\gamma) - \frac{2\gamma}{\gamma^2-\frac{n^2}{4}}\right)\langle\vec{q}|n,\gamma\rangle\ , \tag{59}$$

and the Möbius form corresponds to the kernel obtained by the transformation (20), which means

$$\omega(n,1-\gamma) \to \omega(n,1-\gamma) - \frac{\alpha_s^2 N_c}{4\pi^2}\beta_0\chi'(n,\gamma)\ , \tag{60}$$

we come to the conclusion that in (52) the β-dependent terms in $\omega_M(n,\gamma)$ can be written in the form

$$\frac{\alpha_s^2 N_c}{4\pi^2}\beta_0\left[\chi(n,\gamma)\left(\frac{\partial}{\partial\gamma}-\ln\left(\frac{4e^{-2C}}{\mu^2}\right)-\frac{2\gamma}{\gamma^2-\frac{n^2}{4}}\right)+\frac{\chi^2(n,\gamma)}{2}-\frac{\chi'(n,\gamma)}{2}\right], \tag{61}$$

which is exactly the same as in Eq. (53).

5 SUSY Yang-Mills forward kernel

The extension of the NLO BFKL kernel to supersymmetric theories was performed in Ref. [18] for the forward case in the momentum representation and in Ref. [19] for the nonforward case in the Möbius coordinate representation. Supersymmetric Yang-Mills theories contain gluons and n_M Majorana fermions in the adjoint representation of the color group. For N–extended SUSY we have $n_M = N$. For $N > 1$ besides fermions there are n_S scalar particles; $n_S = 2$ at $N = 2$ and $n_S = 6$ at $N = 4$. Following Ref. [19] for the Möbius kernel in the SUSY theories we write

$$g_{SUSY} = g_{Gluon} + g_{Fermion} + g_{Scalar}. \tag{62}$$

$$g^0_{SUSY}(\vec r_1,\vec r_2;\vec\rho) = -g_{SUSY}(\vec r_1,\vec r_2;\vec\rho) + 2\pi\zeta\,(3)\,\delta\,(\vec\rho)\ , \tag{63}$$

$$\begin{aligned} g_{SUSY}(\vec r_1,\vec r_2;\vec r_2^{\,\prime}) \ &= \frac{\vec r_{12}^{\,2}}{\vec r_{22'}^{\,2}\vec r_{12'}^{\,2}}\left[\frac{67}{18}-\zeta(2)-\frac{5n_M+2n_S}{9}+\frac{\beta_0}{2N_c}\ln\left(\frac{\vec r_{12}^{\,2}\mu^2}{4e^{-2C}}\right)\right. \\ &+\frac{\beta_0}{2N_c}\frac{\vec r_{12'}^{\,2}-\vec r_{22'}^{\,2}}{\vec r_{12}^{\,2}}\ln\left(\frac{\vec r_{22'}^{\,2}}{\vec r_{12'}^{\,2}}\right) \\ &\left.-\frac{1}{2}\ln\left(\frac{\vec r_{12}^{\,2}}{\vec r_{22'}^{\,2}}\right)\ln\left(\frac{\vec r_{12}^{\,2}}{\vec r_{12'}^{\,2}}\right)+\frac{\vec r_{12'}^{\,2}}{2\vec r_{12}^{\,2}}\ln\left(\frac{\vec r_{12'}^{\,2}}{\vec r_{22'}^{\,2}}\right)\ln\left(\frac{\vec r_{12}^{\,2}}{\vec r_{12'}^{\,2}}\right)\right], \end{aligned} \tag{64}$$

$$\begin{aligned} &g_{SUSY}(\vec r_1,\vec r_2;\vec r_1^{\,\prime},\vec r_2^{\,\prime}) = \\ &= \frac{1}{2\vec r_{1'2'}^{\,4}}\left(\frac{\vec r_{11'}^{\,2}\vec r_{22'}^{\,2}-2\vec r_{12}^{\,2}\vec r_{1'2'}^{\,2}}{d}\ln\left(\frac{\vec r_{12'}^{\,2}\vec r_{21'}^{\,2}}{\vec r_{11'}^{\,2}\vec r_{22'}^{\,2}}\right)-1\right)\left(1-n_M+\frac{n_S}{2}\right) \\ &+\left(\frac{(2n_S-3n_M)}{4\vec r_{1'2'}^{\,2}}\frac{\vec r_{12}^{\,2}}{d}+\frac{1}{4\vec r_{11'}^{\,2}\vec r_{22'}^{\,2}}\left(\frac{\vec r_{12}^{\,4}}{d}-\frac{\vec r_{12}^{\,2}}{\vec r_{1'2'}^{\,2}}\right)\right)\ln\left(\frac{\vec r_{12'}^{\,2}\vec r_{21'}^{\,2}}{\vec r_{11'}^{\,2}\vec r_{22'}^{\,2}}\right) \\ &+\frac{\ln\left(\frac{\vec r_{12}^{\,2}}{\vec r_{1'2'}^{\,2}}\right)}{4\vec r_{11'}^{\,2}\vec r_{22'}^{\,2}}+\frac{\ln\left(\frac{\vec r_{12}^{\,2}\vec r_{1'2'}^{\,2}}{\vec r_{11'}^{\,2}\vec r_{22'}^{\,2}}\right)}{2\vec r_{12'}^{\,2}\vec r_{21'}^{\,2}}\left(\frac{\vec r_{12}^{\,2}}{2\vec r_{1'2'}^{\,2}}+\frac{1}{2}-\frac{\vec r_{22'}^{\,2}}{\vec r_{1'2'}^{\,2}}\right)+\frac{\vec r_{12}^{\,2}\ln\left(\frac{\vec r_{12}^{\,2}\vec r_{1'2'}^{\,2}}{\vec r_{12'}^{\,2}\vec r_{21'}^{\,2}}\right)}{4\vec r_{11'}^{\,2}\vec r_{22'}^{\,2}\vec r_{1'2'}^{\,2}} \end{aligned}$$

$$+\frac{\ln\left(\frac{\vec r^{\,2}_{22'}}{\vec r^{\,2}_{12}}\right)}{2\vec r^{\,2}_{11'}\,\vec r^{\,2}_{12'}}+\frac{\ln\left(\frac{\vec r^{\,2}_{12}\vec r^{\,2}_{1'2'}}{\vec r^{\,2}_{12'}\vec r^{\,2}_{22'}}\right)}{2\vec r^{\,2}_{11'}\,\vec r^{\,2}_{1'2'}}+\frac{\ln\left(\frac{\vec r^{\,2}_{12}\vec r^{\,2}_{11'}}{\vec r^{\,2}_{22'}\,\vec r^{\,2}_{1'2'}}\right)}{2\vec r^{\,2}_{12'}\vec r^{\,2}_{1'2'}}+\frac{\vec r^{\,2}_{12}\ln\left(\frac{\vec r^{\,2}_{11'}}{\vec r^{\,2}_{1'2'}}\right)}{2\vec r^{\,2}_{11'}\vec r^{\,2}_{12'}\,\vec r^{\,2}_{22'}}$$

$$+\frac{\vec r^{\,2}_{21'}\ln\left(\frac{\vec r^{\,2}_{21'}\vec r^{\,2}_{1'2'}}{\vec r^{\,2}_{12}\,\vec r^{\,2}_{11'}}\right)}{2\vec r^{\,2}_{11'}\vec r^{\,2}_{22'}\vec r^{\,2}_{1'2'}}+(1\leftrightarrow 2,1'\leftrightarrow 2'),\quad d=\vec r^{2}_{12'}\vec r^{2}_{21'}-\vec r^{2}_{11'}\vec r^{2}_{22'}. \qquad (65)$$

One should insert these functions g into the definition of the Möbius kernel in the coordinate representation (6). Using the results of the previous section we can write the forward Möbius kernel in the SUSY case. It reads

$$\langle\vec r|\hat K_M^{SUSY}|\vec r\,'\rangle=\frac{\alpha_s(\frac{4e^{-2C}}{\vec r^{\,2}})N_c}{2\pi^2}\int\frac{d\vec\rho\,\vec r^{\,2}}{(\vec r-\vec\rho)^2\vec\rho^{\,2}}\left\{\left(2\delta(\vec\rho-\vec r\,')-\delta(\vec r-\vec r\,')\right)\right.$$

$$\times\left[1+\frac{\alpha_sN_c}{4\pi}\left(\frac{67}{9}-2\zeta(2)-\frac{10n_M}{9}-\frac{4n_S}{9}+\frac{\beta_0}{N_c}\frac{\vec\rho^{\,2}-(\vec r-\vec\rho)^2}{\vec r^{\,2}}\ln\left(\frac{(\vec r-\vec\rho)^2}{\vec\rho^{\,2}}\right)\right)\right]$$

$$\left.+\frac{\alpha_sN_c}{4\pi}3\delta(\vec r-\vec r\,')\ln\left(\frac{\vec\rho^{\,2}}{\vec r^{\,2}}\right)\ln\left(\frac{(\vec r-\vec\rho)^2}{\vec r^{\,2}}\right)\right\}$$

$$+\frac{\alpha_s^2N_c^2}{4\pi^3}\frac{\vec r^{\,2}}{\vec r\,'^2}\left(f_1(\vec r,\vec r\,')+f_2^{SUSY}(\vec r,\vec r\,')-\frac{1}{(\vec r-\vec r\,')^2}\ln^2\left(\frac{\vec r^{\,2}}{\vec r\,'^2}\right)\right), \qquad (66)$$

where

$$f_2^{SUSY}(\vec r,\vec r\,')=(1-n_M+\frac{n_S}{2})f_2(\vec r,\vec r\,')+(2n_S-3n_M)\int_0^\infty dt\frac{\ln\left|\frac{1+t}{1-t}\right|}{\vec r\,'^2+t^2\vec r^{\,2}}\,, \qquad (67)$$

the functions f_1 and f_2 are defined in Eqs. (36), (37),

$$\alpha_s(\frac{4e^{-2C}}{\vec r^{\,2}})\simeq\alpha_s(\mu^2)\left(1-\frac{\alpha_s(\mu^2)}{4\pi}\beta_0\ln\left(\frac{4e^{-2C}}{\vec r^{\,2}\mu^2}\right)\right),$$

$$\beta_0=\left(\frac{11}{3}-\frac{2n_M}{3}-\frac{n_S}{6}\right)N_c\,, \qquad (68)$$

μ being the renormalization scale in the $\overline{MS}$-scheme.

As it is known, at $N=4$ the coupling α_s does not run, so that $\beta_0=0$. Moreover, it is seen from (67) that $f_2^{SUSY}=0$ in this case. Next, let us express our result in the renormalization scheme which preserves the supersymmetry. This scheme is known as the dimensional reduction and it differs from the $\overline{MS}$-scheme in the finite charge renormalization (see [19]) for details)

$$\alpha_s\to\alpha_s\left(1-\frac{\alpha_sN_c}{12\pi}\right). \qquad (69)$$

Finally, in the $N = 4$ case, having $\beta_0 = 0$, $n_S = 6$, $n_M = 4$, the kernel simplifies to

$$\langle\vec{r}|\hat{K}_M^{N=4}|\vec{r}\,'\rangle = \frac{\alpha_s N_c}{2\pi^2}\int \frac{d\vec{\rho}\,\vec{r}^{\,2}}{(\vec{r}-\vec{\rho})^2\vec{\rho}^{\,2}}\left(2\delta(\vec{\rho}-\vec{r}\,')-\delta(\vec{r}-\vec{r}\,')\right)\left[1-\frac{\alpha_s N_c}{2\pi}\zeta(2)\right]$$

$$+\frac{\alpha_s^2 N_c^2}{4\pi^3}\left[6\pi\zeta(3)\delta(\vec{r}-\vec{r}\,')+\frac{\vec{r}^{\,2}}{\vec{r}\,'^2}\left(f_1(\vec{r},\vec{r}\,')-\frac{1}{(\vec{r}-\vec{r}\,')^2}\ln^2\left(\frac{\vec{r}^{\,2}}{\vec{r}\,'^2}\right)\right)\right]\ . \quad (70)$$

Now let us consider the forward BFKL kernel in the momentum space. The explicit form of the kernel can be found for QCD in Ref. [3] and for in N=4 Yang-Mills theory in Ref. [18]. In both cases the kernel is written in the space-time dimension $D = 4+2\epsilon$ to regularize infrared divergencies. Here we solve two problems. First, we found the explicit form of the kernel for SUSY Yang-Mills with any N. Second, we perform explicitly the cancellation of the infrared divergencies and write the kernel at physical space-time dimension $D = 4$. It permits us to demonstrate that the functional identity of the forward BFKL kernels in the momentum and Möbius coordinate representations exhibited in the previous section in the LO is preserved in the NLO in the N=4 SUSY case.

To solve the first problem we have to change the quark contribution in the kernel Ref. [3] for the gluino one and to add the scalar particle contribution. It is known (see Refs. [18, 19]) that the gluino contribution to the "real" kernel can be obtained from the quark one by the change of the coefficients $n_f \to n_M N_c$ for the "non-Abelian" (leading at large N_c) part (including the trajectory) and $n_f \to -n_M N_c^3$ for the "Abelian" (suppressed at large N_c) part, so that this contribution is found quite easy. To obtain the scalar contribution is a more subtle task. According to Ref. [19], the scalar contribution also can be divided into "non-Abelian" and "Abelian" parts. Both in the integral representation of the trajectory and in the "non-Abelian" part of the "real" kernel the scalar contribution can be obtained from the corresponding quark contribution by the substitution $n_f \to n_S N_c$ and the change of the fermion polarization operator for the scalar one which differs from the former by the factor $1/(4(1+\epsilon))$. We are interested in the kernel expanded in powers of ϵ. It is clear that since the factor mentioned above depends on ϵ this kernel can be obtained by the substitutions, which are different for different terms in the expansion of the polarization operator. It is not difficult to see that the substitutions must be

$$\frac{2}{3}\frac{n_f}{N_c} \to \frac{n_S}{6},\quad -\frac{10}{9}\frac{n_f}{N_c} \to -\frac{4n_S}{9},\quad \frac{56}{27}\frac{n_f}{N_c} \to \frac{26n_S}{27}. \quad (71)$$

The n-th equality here corresponds to the n-th term in the expansion.

Therefore from Eq. (6) of Ref. [3] we obtain for the trajectory (in the $\overline{MS}$ scheme)

$$\omega\left(-\vec{q}^{\,2}\right)=-\bar{g}_\mu^2\left(\frac{2}{\epsilon}+2\ln\frac{\vec{q}^{\,2}}{\mu^2}\right)-\bar{g}_\mu^4\left[\frac{\beta_0}{N_c}\left(\frac{1}{\epsilon^2}-\ln^2\left(\frac{\vec{q}^{\,2}}{\mu^2}\right)\right)\right.$$

$$+\left(\frac{67}{9}-2\zeta\left(2\right)-\frac{10}{9}n_M-\frac{4n_S}{9}\right)\left(\frac{1}{\epsilon}+2\ln\left(\frac{\vec{q}^{\,2}}{\mu^2}\right)\right)$$

$$\left.-\frac{404}{27}+2\zeta(3)+\frac{56}{27}n_M+\frac{26}{27}n_S\right], \tag{72}$$

where

$$g=g_\mu\mu^{-\epsilon}\left[1+\frac{\beta_0}{N_c}\frac{\bar{g}_\mu^2}{2\epsilon}\right], \quad \bar{g}_\mu^2=\frac{g_\mu^2N_c\Gamma(1-\epsilon)}{(4\pi)^{2+\epsilon}}, \quad \beta_0=\left(\frac{11}{3}-\frac{2}{3}n_M-\frac{n_S}{6}\right). \tag{73}$$

The "Abelian" part of the scalar contribution to the non-forward kernel is given by Eq. (28) of Ref. [19]. For the forward case, using Feynman parametrization and performing integration over k_1, we obtain at $D=4$ (for details see Ref. [25])

$$\langle\vec{q}|\hat{\mathcal{K}}_a^S|\vec{q}^{\,\prime}\rangle=\frac{\alpha_s^2N_c^2}{4\pi^3}\frac{n_s}{2}\frac{1}{16\vec{q}^{\,4}}\left\{\left(3(\vec{q}\vec{q}^{\,\prime})^2-2\vec{q}^{\,2}\vec{q}^{\,\prime 2}\right)\right.$$

$$\times\left(\frac{2}{\vec{q}^{\,2}}+\frac{2}{\vec{q}^{\,\prime 2}}-\left(\frac{2}{\vec{q}^{\,2}}-\frac{2}{\vec{q}^{\,\prime 2}}\right)\ln\left(\frac{\vec{q}^{\,2}}{\vec{q}^{\,\prime 2}}\right)\right)$$

$$\left.+\left[(\vec{q}\vec{q}^{\,\prime})^2\left(2-3\frac{\vec{q}^{\,\prime 2}}{\vec{q}^{\,2}}-3\frac{\vec{q}^{\,2}}{\vec{q}^{\,\prime 2}}\right)+2\left(\vec{q}^{\,2}+\vec{q}^{\,\prime 2}\right)^2\right]\int\limits_0^\infty\frac{dt}{\vec{q}^{\,2}+t^2\vec{q}^{\,\prime 2}}\ln\left|\frac{1+t}{1-t}\right|\right\}$$

$$=\frac{\alpha_s^2N_c^2}{4\pi^3}\frac{n_s}{2}\frac{\vec{q}^{\,\prime 2}}{\vec{q}^{\,2}}\left(f_2(\vec{q},\vec{q}^{\,\prime})+4\int\limits_0^\infty\frac{dt}{\vec{q}^{\,2}+t^2\vec{q}^{\,\prime 2}}\ln\left|\frac{1+t}{1-t}\right|\right), \tag{74}$$

where f_2 is defined in (37). Taking into account that $\langle\vec{q}|\hat{\mathcal{K}}|\vec{q}^{\,\prime}\rangle$ differs from the symmetric kernel

$$K(\vec{q},\vec{q}^{\,\prime})=K_r(\vec{q},\vec{q}^{\,\prime})+2\delta(\vec{q}-\vec{q}^{\,\prime})\omega\left(-\vec{q}^{\,2}\right) \tag{75}$$

presented in Ref. [3] by the factor $\vec q\,'^2/\vec q\,^2$, we obtain for the "real" part

$$K_r(\vec q,\vec q\,') = \frac{4\,\overline{g}^2_\mu\,\mu^{-2\epsilon}}{\pi^{1+\epsilon}\Gamma(1-\epsilon)}\frac{1}{\vec k\,^2} + \frac{4\overline{g}^4_\mu\,\mu^{-2\epsilon}}{\pi^{1+\epsilon}\Gamma(1-\epsilon)}$$

$$\times\left\{\frac{1}{\vec k\,^2}\left[\frac{\beta_0}{N_c\epsilon}\left(1-\left(\frac{\vec k\,^2}{\mu^2}\right)^\epsilon(1-\epsilon^2\frac{\pi^2}{6})\right)+\left(\frac{\vec k\,^2}{\mu^2}\right)^\epsilon\left(\frac{67}{9}-\frac{\pi^2}{3}-\frac{10}{9}n_M-\frac{4n_S}{9}\right.\right.\right.$$

$$\left.\left.+\epsilon\left(-\frac{404}{27}+\frac{11}{3}\zeta(2)+14\zeta(3)+\frac{56}{27}n_M+\frac{26}{27}n_S\right)-\ln^2\frac{\vec q\,^2}{\vec q\,'^2}\right)\right]+f_1\left(\vec q_1,\vec q_1\,'\right)$$

$$\left.+f_2^{SUSY}\left(\vec q_1,\vec q_1\,'\right)\right\}, \qquad (76)$$

where $\vec k = \vec q - \vec q\,'$, f_1 is defined in (37) and f_2^{SUSY} in (67).

Eqs. (72) and (76) solve the first problem raised in the beginning of this section. But they contain the infrared divergencies which complicate their use. One can cancel the divergencies and write the kernel at physical space-time dimension $D=4$ following Refs. [14, 17] and using the integral representation for the trajectory

$$\omega(-\vec q\,^2) = -\frac{\bar g^2_\mu\,\vec q\,^2}{\pi^{1+\epsilon}\Gamma(1-\epsilon)}\int\frac{d^{2+2\epsilon}k\ \mu^{-2\epsilon}}{\vec k\,^2(\vec k-\vec q)^2}\left(1+\bar g^2_\mu f_\omega(\vec k,\vec k-\vec q)\right)\ . \qquad (77)$$

The quark contribution to f_ω is defined by Eqs. (76), (77) of Ref. [14]. The gluino and scalar contributions can be obtained from the quark one by the substitutions discussed above. The gluon contribution is given, with the required accuracy, by Eq. (23) of Ref. [17]. As the result, we obtain with this accuracy

$$f_\omega(\vec k_1,\vec k_2) = \frac{\beta_0}{N_c\epsilon}+\left[\frac{\beta_0}{N_c\epsilon}-\frac{67}{9}+2\zeta\,(2)+\frac{10}{9}n_M+\frac{4n_S}{9}\right.$$

$$\left.+\,\epsilon\left(\frac{404}{27}-\frac{11}{3}\zeta(2)-6\zeta(3)-\frac{56}{27}n_M-\frac{26}{27}n_S\right)\right]$$

$$\times\left[\left(\frac{\vec k^{\,2}_{12}}{\mu^2}\right)^\epsilon-\left(\frac{\vec k^{\,2}_1}{\mu^2}\right)^\epsilon-\left(\frac{\vec k^{\,2}_2}{\mu^2}\right)^\epsilon\right]-\ln\left(\frac{\vec k^{\,2}_{12}}{\vec k^{\,2}_1}\right)\ln\left(\frac{\vec k^{\,2}_{12}}{\vec k^{\,2}_2}\right). \qquad (78)$$

Just like Refs. [14] and [17], in the limit $\epsilon\to0$ we introduce the cut-off $\lambda\to0$ keeping $\epsilon\ln\lambda\to0$. Then in the regions $\vec k\,^2\le\lambda^2$ we have

$$f_\omega(\vec{k},\vec{k}-\vec{q}) = \frac{\beta_0}{N_c\epsilon} - \left(\frac{\vec{k}^{\,2}}{\mu^2}\right)^\epsilon \left[\frac{\beta_0}{N_c\epsilon} - \frac{67}{9} + 2\zeta\,(2) + \frac{10}{9}n_M + \frac{4n_S}{9}\right.$$

$$\left. +\epsilon\left(\frac{404}{27} - \frac{11}{3}\zeta(2) - 6\zeta(3) - \frac{56}{27}n_M - \frac{26}{27}n_S\right)\right] \tag{79}$$

and in the region $(\vec{k}-\vec{q})^2 \le \lambda^2$ the same expression with the substitution $\vec{k}^{\,2} \to (\vec{k}-\vec{q})^2$. Comparing (79) with (76) we see that when the kernel $K(\vec{q},\vec{q}\,')$ (75) acts on any nonsingular at $\vec{q}=\vec{q}\,'$ function the contribution of the region $\vec{k}^{\,2} \le \lambda^2$ in the the "real" part cancels almost completely the contributions of the regions $\vec{k}^{\,2} \le \lambda^2$ and $(\vec{k}-\vec{q})^2 \le \lambda^2$ in the doubled trajectory $\omega(-\vec{q}^{\,2})$. The only piece which remains uncancelled is

$$2\frac{\bar{g}_\mu^4}{\pi^{1+\epsilon}\Gamma(1-\epsilon)}\int\frac{d^{2+2\epsilon}k\;\mu^{-2\epsilon}}{\vec{k}^{\,2}}16\epsilon\zeta(3)\left(\frac{\vec{k}^{\,2}}{\mu^2}\right)^\epsilon\theta(\lambda^2-\vec{k}^{\,2}) = 2\frac{\alpha_s^2(\mu)N_c^2}{2\pi^2}\zeta(3). \tag{80}$$

Outside the regions $\vec{k}^{\,2} \le \lambda^2$ and $(\vec{k}-\vec{q})^2 \le \lambda^2$ one can put $\epsilon = 0$. Thus we come to the representation of the symmetric kernel

$$K(\vec{q},\vec{q}\,') = \frac{\alpha_s(\vec{q}^{\,2})N_c}{2\pi^2}\left[\frac{2}{(\vec{q}-\vec{q}\,')^2} - \delta(\vec{q}-\vec{q}\,')\int\frac{d\vec{l}\,\vec{q}^{\,2}}{(\vec{q}-\vec{l})^2\vec{l}^{\,2}}\right]$$

$$\times\left[1+\frac{\alpha_s N_c}{4\pi}\left(\frac{67}{9} - 2\zeta\,(2) - \frac{10}{9}n_M - \frac{4n_S}{9}\right)\right]$$

$$+\frac{\alpha_s^2 N_c^2}{4\pi^3}\left[\frac{1}{(\vec{q}-\vec{q}\,')^2}\frac{\beta_0}{N_c}\ln\left(\frac{\vec{q}^{\,2}}{(\vec{q}-\vec{q}\,')^{\,2}}\right)\right.$$

$$+f_1(\vec{q},\vec{q}\,') + f_2^{SUSY}(\vec{q},\vec{q}\,') - \frac{1}{(\vec{q}-\vec{q}\,')^2}\ln^2\left(\frac{\vec{q}^{\,2}}{\vec{q}\,'^2}\right)$$

$$\left.+\delta(\vec{q}-\vec{q}\,')\left(\frac{\beta_0}{2N_c}\int\frac{d\vec{l}\,\vec{q}^{\,2}}{(\vec{q}-\vec{l})^2\vec{l}^{\,2}}\ln\left(\frac{(\vec{q}-\vec{l})^2\vec{l}^{\,2}}{\vec{q}^{\,4}}\right) + 6\pi\zeta(3)\right)\right]\;, \tag{81}$$

which solves the second problem: presentation of the kernel in the physical space-time dimension $D = 4$ with explicit cancellation of the infrared divergencies.

To compare the BFKL kernel in the Möbius representation (66) and in the momentum representation, we have to take into account that (66) corresponds to the kernel obtained from the symmetric one by the transformation (20) with $\hat{\vec{q}}_1^{\,2} = \hat{\vec{q}}_2^{\,2} = \hat{\vec{q}}^{\,2}$ and that $\langle \vec{q}|\hat{\mathcal{K}}|\vec{q}\,'\rangle = \vec{q}\,'^{2} K(\vec{q},\vec{q}\,')\vec{q}^{\,-2}$, so that

$$\langle \vec{q}|\hat{\mathcal{K}}|\vec{q}\,'\rangle = \frac{\alpha_s(\vec{q}^{\,2})N_c}{2\pi^2}\int \frac{d\vec{l}\,\vec{q}\,'^{2}}{(\vec{q}-\vec{l})^2\vec{l}^{\,2}}\left[2\delta(\vec{q}-\vec{l}) - \delta(\vec{q}-\vec{q}\,')\right]$$

$$\times\left[1+\frac{\alpha_s N_c}{4\pi}\left(\frac{67}{9}-2\zeta\left(2\right)-\frac{10}{9}n_M-\frac{4n_S}{9}\right)\right]$$

$$+\frac{\alpha_s^2 N_c^2}{4\pi^3}\frac{\vec{q}\,'^{2}}{\vec{q}^{\,2}}\left[\frac{1}{(\vec{q}-\vec{q}\,')^2}\frac{\beta_0}{N_c}\ln\left(\frac{\vec{q}^{\,2}}{(\vec{q}-\vec{q}\,')^{\,2}}\right)\right.$$

$$+f_1(\vec{q},\vec{q}\,') + f_2^{SUSY}(\vec{q},\vec{q}\,') - \frac{1}{(\vec{q}-\vec{q}\,')^2}\ln^2\left(\frac{\vec{q}^{\,2}}{\vec{q}\,'^2}\right)$$

$$\left.+\delta(\vec{q}-\vec{q}\,')\left(\frac{\beta_0}{2N_c}\int\frac{d\vec{l}\,\vec{q}^{\,2}}{(\vec{q}-\vec{l})^2\vec{l}^{\,2}}\ln\left(\frac{(\vec{q}-\vec{l})^2\vec{l}^{\,2}}{\vec{q}^{\,4}}\right)+6\pi\zeta(3)\right)\right]\,. \qquad (82)$$

Comparing this expression with Eq. (66) one can see that at $\beta = 0$ they are functionally identical up to the normalization factors:

$$\frac{\vec{r}\,'^{2}}{\vec{r}^{\,2}}\langle \vec{r}|\hat{\mathcal{K}}_M|\vec{r}^{\,2}\rangle|_{\beta_0=0} = \frac{\vec{q}^{\,2}}{\vec{q}\,'^2}\langle\vec{q}|\hat{\mathcal{K}}|\vec{q}\,'\rangle|_{\beta_0=0}\Bigg|_{\vec{q}\to\vec{r},\ \vec{q}\,'\to\vec{r}\,'}\,. \qquad (83)$$

Note that with these normalization factors the kernels are symmetric with respect to $\vec{r}\leftrightarrow\vec{r}\,'$ or $\vec{q}\leftrightarrow\vec{q}\,'$ substitution.

Actually the identity relation can be expected, because at $\beta = 0$ the kernels $\hat{\vec{r}}^{\,-2}\hat{\mathcal{K}}_M\hat{\vec{r}}^{\,2}$ and $\hat{\vec{q}}^{\,2}\hat{\mathcal{K}}\hat{\vec{q}}^{\,-2}$ have the same eigenvalues corresponding to the eigenfunctions connected by the replacement $\vec{r}\leftrightarrow\vec{q}$.

6 Comparison of the kernels: nonforward case

Here we present the results of the simplest generalizations for the nonforward case the transformation used in section 4 for the elimination of the discrepancy between the results of Refs. [17] and [12] for the forward scattering. We start from the commutator $\left[\hat{\mathcal{K}}^{(B)},\,\hat{\mathcal{K}}^{(B)}\frac{1}{4}\ln\left(\hat{q}_1^2\hat{q}_2^2\right)\right]$. The calculation of this commutator is described in the appendix B. The result is

$$\left(\frac{2\pi^2}{\alpha_s N_c}\right)^2 \langle\vec{r}_1,\vec{r}_2|\left[\hat{\mathcal{K}}^{(B)},\,\hat{\mathcal{K}}^{(B)}\frac{1}{4}\ln\left(\hat{q}_1^2\hat{q}_2^2\right)\right]_M|\vec{r}_1^{\,\prime},\vec{r}_2^{\,\prime}\rangle =$$

$$= \frac{1}{2}\int d\vec{\rho}\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{1'2'}^{\,2}\vec{r}_{1\rho}^{\,2}\vec{r}_{2\rho}^{\,2}}\left(\left(\delta\left(\vec{r}_{11'}\right)-\delta\left(\vec{r}_{1'\rho}\right)\right)\frac{\left(\vec{r}_{21'}\,\vec{r}_{1'2'}\right)}{\vec{r}_{22'}^{\,2}}\ln\left(\frac{\vec{r}_{21'}^{\,2}}{\vec{r}_{1'2'}^{\,2}}\right)\right.$$

$$\left.+\delta\left(\vec{r}_{11'}\right)\frac{\left(\vec{r}_{1'2'}\,\vec{r}_{1'\rho}\right)}{\vec{r}_{2'\rho}^{\,2}}\ln\left(\frac{\vec{r}_{1'\rho}^{\,2}}{\vec{r}_{1'2'}^{\,2}}\right)\right)$$

$$+\vec{r}_{12}^{\,2}\left(\frac{\ln\left(\frac{\vec{r}_{12'}^{\,2}\vec{r}_{21'}^{\,2}}{\vec{r}_{12}^{\,2}\vec{r}_{1'2'}^{\,2}}\right)}{4\vec{r}_{11'}^{\,2}\vec{r}_{22'}^{\,2}\vec{r}_{1'2'}^{\,2}}+\frac{\ln\left(\frac{\vec{r}_{11'}^{\,2}\vec{r}_{22'}^{\,2}}{\vec{r}_{12}^{\,2}\vec{r}_{1'2'}^{\,2}}\right)}{4\vec{r}_{11'}^{\,2}\vec{r}_{21'}^{\,2}}\left(\frac{1}{\vec{r}_{1'2'}^{\,2}}-\frac{1}{\vec{r}_{22'}^{\,2}}\right)\right.$$

$$\left.+\frac{\ln\left(\frac{\vec{r}_{11'}^{\,2}\vec{r}_{22'}^{\,2}}{\vec{r}_{12'}^{\,2}\vec{r}_{21'}^{\,2}}\right)}{4\vec{r}_{11'}^{\,2}\vec{r}_{1'2'}^{\,2}}\left(\frac{\vec{r}_{12'}^{\,2}}{\vec{r}_{22'}^{\,2}}-1\right)\right)+(1\leftrightarrow 2, 1'\leftrightarrow 2'). \qquad (84)$$

The subscript M means that the operator acts in the Möbius representation, i.e. the matrix element (84) vanishes as $\vec{r}_1 \to \vec{r}_2$. One can see that the above expression has ultraviolet singularities which cancel in the convolution with Möbius impact factors. But the structure of these singularities is different from the structure of singularities in the kernel (7), (8). Therefore it is obvious that this commutator cannot make g^0 and $g(\vec{r}_1,\vec{r}_2,\vec{r}_2^{\,\prime})$ coincident with the corresponding g^0_{BC} and $g^0_{BC}(\vec{r}_1,\vec{r}_2,\vec{r}_2^{\,\prime})$. As for $g(\vec{r}_1,\vec{r}_2;\vec{r}_1^{\,\prime},\vec{r}_2^{\,\prime})$, we have

$$g(\vec{r}_1,\vec{r}_2;\vec{r}_1^{\,\prime},\vec{r}_2^{\,\prime})-g_{BC}(\vec{r}_1,\vec{r}_2;\vec{r}_1^{\,\prime},\vec{r}_2^{\,\prime}) = \frac{\ln\left(\frac{\vec{r}^{\,2}_{12}}{\vec{r}^{\,2}_{1'2'}}\right)}{4\vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{22'}}+\frac{\ln\left(\frac{\vec{r}^{\,2}_{12}\vec{r}^{\,2}_{1'2'}}{\vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{22'}}\right)}{4\vec{r}^{\,2}_{12'}\,\vec{r}^{\,2}_{21'}}$$

$$-\frac{\ln\left(\frac{\vec{r}^{\,2}_{12'}\vec{r}^{\,2}_{21'}}{\vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{22'}}\right)}{4\vec{r}^{\,4}_{1'2'}}+\frac{\ln\left(\frac{\vec{r}^{\,2}_{22'}}{\vec{r}^{\,2}_{12}}\right)}{2\vec{r}^{\,2}_{11'}\,\vec{r}^{\,2}_{12'}}+\frac{\ln\left(\frac{\vec{r}^{\,2}_{12}\vec{r}^{\,2}_{1'2'}}{\vec{r}^{\,2}_{12'}\vec{r}^{\,2}_{22'}}\right)}{2\vec{r}^{\,2}_{11'}\,\vec{r}^{\,2}_{1'2'}}+\frac{\ln\left(\frac{\vec{r}^{\,2}_{12}\vec{r}^{\,2}_{11'}}{\vec{r}^{\,2}_{22'}\,\vec{r}^{\,2}_{1'2'}}\right)}{2\vec{r}^{\,2}_{12'}\vec{r}^{\,2}_{1'2'}}$$

$$+\frac{\vec{r}^{\,2}_{22'}\ln\left(\frac{\vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{22'}}{\vec{r}^{\,2}_{12}\,\vec{r}^{\,2}_{1'2'}}\right)}{2\vec{r}^{\,2}_{12'}\vec{r}^{\,2}_{21'}\vec{r}^{\,2}_{1'2'}}+\frac{\vec{r}^{\,2}_{21'}\ln\left(\frac{\vec{r}^{\,2}_{21'}\vec{r}^{\,2}_{1'2'}}{\vec{r}^{\,2}_{12}\,\vec{r}^{\,2}_{11'}}\right)}{2\vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{22'}\vec{r}^{\,2}_{1'2'}}+\frac{\vec{r}^{\,2}_{12}\ln\left(\frac{\vec{r}^{\,2}_{11'}}{\vec{r}^{\,2}_{1'2'}}\right)}{2\vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{12'}\,\vec{r}^{\,2}_{22'}}$$

$$+\frac{\vec{r}^{\,2}_{12}\ln\left(\frac{\vec{r}^{\,2}_{12}\vec{r}^{\,2}_{1'2'}}{\vec{r}^{\,2}_{12'}\,\vec{r}^{\,2}_{21'}}\right)}{4\vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{22'}\vec{r}^{\,2}_{1'2'}}+\frac{\vec{r}^{\,2}_{12}\ln\left(\frac{\vec{r}^{\,2}_{12}\,\vec{r}^{\,2}_{1'2'}}{\vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{22'}}\right)}{4\vec{r}^{\,2}_{12'}\vec{r}^{\,2}_{21'}\,\vec{r}^{\,2}_{1'2'}}+(1\leftrightarrow 2, 1'\leftrightarrow 2') \qquad (85)$$

and again we see that this commutator does not help to eliminate the discrepancy.

Then one can search for a commutator equal to $\left[\hat{\mathcal{K}}^{(B)},\,\hat{\mathcal{K}}^{(B)}\frac{1}{4}\ln\left(\hat{q}_1^2\hat{q}_2^2\right)\right]_M$ in the forward case but different from it in general. An example of this kind is $\left[\hat{\mathcal{K}}^{(B)},\,\frac{1}{4}\ln\left(\hat{q}_1^2\hat{q}_2^2\right)\hat{\mathcal{K}}^{B}\right]_M$. Indeed, this commutator coincides with the previous one in the forward case since acting on the eigenfunctions $\langle\vec{q}|n,\gamma\rangle \propto e^{in\phi_{\vec{q}}}\left(\vec{q}^{\,2}\right)^{\gamma-2}$ they yield the same result:

$$\begin{aligned}\langle\vec{q}|\left[\hat{\mathcal{K}}^{(B)},\,\hat{\mathcal{K}}^{(B)}\ln\left(\hat{q}^2\right)\right]|n,\gamma\rangle &= \int d^2l\,\langle\vec{q}|\hat{\mathcal{K}}^{(B)}\left[\hat{\mathcal{K}}^{(B)},\ln\left(\hat{q}^2\right)\right]|\vec{l}\rangle\langle\vec{l}|n,\gamma\rangle \\ &= \chi\left(n,\gamma\right)\chi'\left(n,\gamma\right)\langle\vec{q}|n,\gamma\rangle \\ &= \langle\vec{q}|\left[\hat{\mathcal{K}}^{(B)},\,\ln\left(\hat{q}^2\right)\hat{\mathcal{K}}^{(B)}\right]|n,\gamma\rangle. \end{aligned} \tag{86}$$

Here we have used the equality $(\vec{q}^{\,2})^\gamma\ln\left(\vec{q}^{\,2}\right) = \partial(\vec{q}^{\,2})^\gamma/(\partial\gamma)$ to perform the integration. The coordinate representation of $\left[\hat{\mathcal{K}}^{(B)},\,\frac{1}{4}\ln\left(\hat{q}_1^2\hat{q}_2^2\right)\hat{\mathcal{K}}^{(B)}\right]_M$ is also given in the appendix B and it also fails to resolve the difference between the BFKL and BK kernels. Thus, the simplest generalizations are unable to eliminate the discrepancy in the nonforward case.

7 Summary

In this paper we studied properties of the next-to-leading BFKL kernel in gluodynamics and SUSY Yang-Mills theories. In particular our study is connected with the discrepancy between the gluon contribution to the BFKL kernel in the Möbius representation found in Ref. [17] and the kernel of the linearized BK equation calculated in Ref. [12].

We analyzed the ambiguity of the next-to-leading kernel, in particular connected with the energy scale. Broadly speaking, the ambiguity is related to rearrangement of the radiative corrections between the kernel and the impact factors, and the one connected with the energy scale is not an exception. It is shown that the change of the energy scale can be associated with the specific form of the general transformation of the NLO kernel discussed in Refs. [14] and [17].

The ambiguity can be used to remove the discrepancy between the results of Refs. [17] and [12]. It was explicitly demonstrated in the case of forward scattering. We found the Möbius kernel for this case and showed that the major part of the difference between this kernel and the corresponding kernel of Ref. [12] can be eliminated by the suitable transformormation, which can be associated with the change of the energy scale, so that the difference is reduced to two terms. One of them is proportional to the first coefficient of the β-function. In our opinion, this term is connected with the difference of the renormalization scheme used in Ref. [12] from conventional $\overline{MS}$-scheme. This term can be eliminated by the change of the renormalization scheme. Unfortunately, we can not eliminate the third term, proportional $\zeta(3)$. In the BFKL approach this term passed through a great number of verifications. It is also confirmed by the calculation of the three-loop anomalous dimensions in Refs. [23] and [24].

We calculated the characteristic function $\omega_M(n, \gamma)$ describing action of the forward Möbius kernel on the eigenfunctions of the leading order kernel and compared it with the corresponding function of the BFKL kernel in the momentum representation found in Ref. [18]. The forward Möbius kernel was found using the results of Refs. [14, 16, 17] whereas the calculation of Ref. [18] is based on the results of Ref. [3]. Therefore the comparison serves as the cross-check of the results of these papers. The coincidence of the characteristic functions gives a strong argument in favour of rightness of the used results.

We studied also the forward BFKL kernel in supersymmetric Yang-Mills theories for any N-extended SUSY both in the momentum and in the Möbius coordinate representations and demonstrated the functional identity (up to the normalization factors) of the form of the kernel in these representations for $N = 4$. We calculated the kernel in the Möbius representation in the impact parameter space using the results of Refs. [14, 16, 17, 19] and found the kernel for any N in the momentum space using the results of Refs. [3, 18]. Performing explicit cancellation of the infrared divergencies and writing the kernel at physical space-time dimension $D = 4$ we demonstrated the functional identity mentioned above, that confirms correctness of used results.

At last, we checked how the simplest generalizations of the transformation, used for the elimination of the discrepancy between the results of Refs. [17] and [12] for the forward scattering, work in the general (nonforward) case. Unfortunately, these generalizations are not effective. Of course, it does not mean that the transformation eliminating the discrepancy (apart from the difference in the renormalization scheme and in the $\zeta(3)$ term) does not exist. Moreover, we hope that it exists; but in this case the generalization from the forward case is more refined than we used.

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Appendix A

In the representation (33) for $\langle \vec r|\hat{\mathcal{K}}|\vec r\,'\rangle$ only the term with $g(\vec r_1,\vec r_2;\vec r_1{}',\vec r_2{}')$ in $\langle \vec r_1,\vec r_2|\hat{\mathcal{K}}|\vec r_1{}'\vec r_2{}'\rangle$ requires integration. Let us introduce

$$L(\vec x,\vec z) = \frac{1}{\pi}\int g(\vec r_1,\vec r_2;\vec r_1{}',\vec r_2{}')\delta(\vec r_{1'2'}-\vec z)d\,^2r_1{}'d\,^2r_2{}' \tag{87}$$

where $\vec r_{12}=\vec x$ and $g(\vec r_1,\vec r_2;\vec r_1{}^2,\vec r_2{}^2)$ is defined in Eq. (12). Denoting

$$f_1\left(\vec x,\vec z\right) = \frac{\vec z^{\,2}}{\vec x^{\,2}}\int\frac{d\vec z_1}{2\pi}\left[\frac{\vec x^4}{(\vec x-\vec z-\vec z_1)^{\,2}\vec z_1^{\,2}-(\vec x-\vec z_1)^{\,2}(\vec z+\vec z_1)^{\,2}}+\frac{\vec x^{\,2}}{\vec z^{\,2}}\right] \times\frac{\ln\left(\frac{(\vec x-\vec z-\vec z_1)^{\,2}\vec z_1^{\,2}}{(\vec x-\vec z_1)^{\,2}(\vec z+\vec z_1)^{\,2}}\right)}{\vec z_1^{\,2}(\vec x-\vec z-\vec z_1)^{\,2}} \tag{88}$$

and

$$f_2\left(\vec x,\vec z\right)=\frac{2}{\vec x^{\,2}\vec z^{\,2}}\int\frac{d\,^2\vec z_1}{2\pi}\left(\frac{\left((\vec x-\vec z-\vec z_1)^{\,2}\vec z_1^{\,2}-2\vec x^{\,2}\vec z^{\,2}\right)\ln\left(\frac{(\vec x-\vec z_1)^{\,2}(\vec z+\vec z_1)^{\,2}}{(\vec x-\vec z-\vec z_1)^{\,2}\vec z_1^{\,2}}\right)}{(\vec x-\vec z_1)^{\,2}(\vec z+\vec z_1)^{\,2}-(\vec x-\vec z-\vec z_1)^{\,2}\vec z_1^{\,2}}-1\right), \tag{89}$$

we have

$$L\left(\vec x,\vec z\right) = \frac{\vec x^{\,2}}{\vec z^{\,2}}f_1\left(\vec z,\vec x\right)+\frac{\vec x^{\,2}}{\vec z^{\,2}}f_2\left(\vec x,\vec z\right)$$
$$+\frac{1}{2\pi}\int d\vec z_1\left[\frac{\ln\left(\frac{\vec x^{\,2}}{\vec z^{\,2}}\right)}{2(\vec x-\vec z-\vec z_1)^{\,2}\vec z_1^{\,2}}+\frac{\ln\left(\frac{(\vec x-\vec z-\vec z_1)^{\,2}\vec z_1^{\,2}}{\vec x^{\,2}\vec z^{\,2}}\right)}{(\vec x-\vec z_1)^{\,2}(\vec z+\vec z_1)^{\,2}}\left(\frac{\vec z_1^{\,2}}{\vec z^{\,2}}-\frac{\vec x^{\,2}}{2\vec z^{\,2}}-\frac{1}{2}\right)\right.$$
$$+\frac{\ln\left(\frac{\vec x^{\,2}\vec z^{\,2}}{(\vec x-\vec z_1)^{\,2}(\vec z+\vec z_1)^{\,2}}\right)\vec x^{\,2}}{2\vec z^{\,2}(\vec x-\vec z-\vec z_1)^{\,2}\vec z_1^{\,2}}+\frac{\ln\left(\frac{\vec x^{\,2}(\vec x-\vec z-\vec z_1)^{\,2}}{\vec z^{\,2}\vec z_1^{\,2}}\right)}{\vec z^{\,2}(\vec x-\vec z_1)^{\,2}}+\frac{\ln\left(\frac{\vec z_1^{\,2}}{\vec x^{\,2}}\right)}{(\vec x-\vec z_1)^{\,2}(\vec x-\vec z-\vec z_1)^{\,2}}$$
$$+\frac{\ln\left(\frac{(\vec x-\vec z-\vec z_1)^{\,2}}{\vec z^{\,2}}\right)\vec x^{\,2}}{(\vec x-\vec z_1)^{\,2}(\vec x-\vec z-\vec z_1)^{\,2}\vec z_1^{\,2}}+\frac{\ln\left(\frac{\vec z^{\,2}(\vec z+\vec z_1)^{\,2}}{\vec x^{\,2}(\vec x-\vec z-\vec z_1)^{\,2}}\right)(\vec z+\vec z_1)^{\,2}}{\vec z^{\,2}(\vec x-\vec z-\vec z_1)^{\,2}\vec z_1^{\,2}}$$
$$\left.+\frac{\ln\left(\frac{\vec x^{\,2}\vec z^{\,2}}{(\vec x-\vec z_1)^{\,2}\vec z_1^{\,2}}\right)}{\vec z^{\,2}(\vec x-\vec z-\vec z_1)^{\,2}}+(\vec z_1\to\vec x-\vec z-\vec z_1)\right]. \tag{90}$$

The function f_1 can be obtained from the integral J_{13} in [26]. It reads:

$$f_1\left(\vec x,\vec z\right) = \frac{\vec z^{\,2}}{\vec x^{\,2}}\int\frac{d\vec z_1}{2\pi}\left[\frac{\vec x^4}{(\vec x-\vec z-\vec z_1)^{\,2}\vec z_1^{\,2}-(\vec x-\vec z_1)^{\,2}(\vec z+\vec z_1)^{\,2}}+\frac{\vec x^{\,2}}{\vec z^{\,2}}\right]$$

$$\times \frac{\ln\left(\frac{(\vec{x}-\vec{z}-\vec{z}_1)^{\,2}\vec{z}_1^{\,2}}{(\vec{x}-\vec{z}_1)^{\,2}(\vec{z}+\vec{z}_1)^{\,2}}\right)}{\vec{z}_1^{\,2}(\vec{x}-\vec{z}-\vec{z}_1)^{\,2}} = \frac{\left(\vec{x}^{\,2}-\vec{z}^{\,2}\right)}{(\vec{x}-\vec{z})^{\,2}(\vec{x}+\vec{z})^{\,2}}\left[\ln\left(\frac{\vec{x}^{\,2}}{\vec{z}^{\,2}}\right)\ln\left(\frac{\vec{x}^{\,2}\vec{z}^{\,2}(\vec{x}-\vec{z})^4}{(\vec{x}^{\,2}+\vec{z}^{\,2})^4}\right)\right.$$

$$\left. +2\,\mathrm{Li}_2\left(-\frac{\vec{z}^{\,2}}{\vec{x}^{\,2}}\right)-2\,\mathrm{Li}_2\left(-\frac{\vec{x}^{\,2}}{\vec{z}^{\,2}}\right)\right]$$

$$-\left(1-\frac{\left(\vec{x}^{\,2}-\vec{z}^{\,2}\right)^{\,2}}{(\vec{x}-\vec{z})^{\,2}(\vec{x}+\vec{z})^{\,2}}\right)\left[\int\limits_0^1-\int\limits_1^\infty\right]\frac{du}{(\vec{x}-\vec{z}u)^{\,2}}\ln\left(\frac{u^{\,2}\vec{z}^{\,2}}{\vec{x}^{\,2}}\right)\ . \tag{91}$$

To find f_2 we use the representation

$$\frac{\ln\left(\frac{ab}{cd}\right)}{ab-cd} = \int\limits_0^1 \frac{du}{((1-u)c+ua)((1-u)b+ud)}\ , \tag{92}$$

which permits us to integrate over $\vec{z}_1$ using Feynman parametrization and dimensional regularization and to get f_2 in the form

$$f_2(\vec{x},\vec{z}) = 4\left(\frac{(\vec{x}\,\vec{z})^{\,2}}{\vec{x}^{\,2}\vec{z}^{\,2}}+2\right)J_2-2J_{\vec{x}}-2J_{\vec{z}}-\frac{3}{2}J_1. \tag{93}$$

Here

$$J_1 = \int \frac{d^{2+2\epsilon}\vec{z}_1}{\pi^{1+\epsilon}\Gamma(1-\epsilon)}\frac{\ln\left(\frac{(\vec{x}-\vec{z}_1)^{\,2}(\vec{z}+\vec{z}_1)^{\,2}}{(\vec{x}-\vec{z}-\vec{z}_1)^{\,2}\vec{z}_1^{\,2}}\right)}{(\vec{x}-\vec{z}_1)^{\,2}(\vec{z}+\vec{z}_1)^{\,2}-(\vec{x}-\vec{z}-\vec{z}_1)^{\,2}\vec{z}_1^{\,2}}$$

$$= \int\limits_0^1 du\int\limits_0^1 dv\frac{1}{u(1-u)\vec{z}^{\,2}+v(1-v)\vec{x}^{\,\,2}} = -2\int\limits_0^\infty dt\frac{\ln\left|\frac{1-t}{1+t}\right|}{\vec{z}^{\,2}+t^{\,2}\vec{x}^{\,2}}, \tag{94}$$

$$J_{\vec{x}} = \int\limits_0^1 du\int\limits_0^1 dv\frac{v(1-v)}{u(1-u)\vec{z}^{\,2}+v(1-v)\vec{x}^{\,\,2}}$$

$$= -\frac{1}{4}\left(\left(1-\frac{\vec{z}^{\,2}}{\vec{x}^{\,2}}\right)\int\limits_0^\infty dt\frac{\ln\left|\frac{1-t}{1+t}\right|}{\vec{z}^{\,2}+t^{\,2}\vec{x}^{\,2}}-\frac{2}{\vec{x}^{\,2}}+\frac{\ln\frac{\vec{x}^{\,2}}{\vec{z}^{\,2}}}{\vec{x}^{\,2}}\right), \tag{95}$$

$$J_2 = \int\limits_0^1 du\int\limits_0^1 dv\frac{u(1-u)v(1-v)}{u(1-u)\vec{z}^{\,2}+v(1-v)\vec{x}^{\,\,2}}$$

$$= -\frac{1}{32}\left[\left(1-\frac{3}{2}\left(\frac{\vec{z}^{\,2}}{\vec{x}^{\,2}}+\frac{\vec{x}^{\,2}}{\vec{z}^{\,2}}\right)\right)\int\limits_0^\infty dt \frac{\ln\left|\frac{1-t}{1+t}\right|}{\vec{z}^{\,2}+t^{\,2}\vec{x}^{\,2}}\right.$$
$$\left. -3\left(\frac{1}{\vec{x}^{\,2}}+\frac{1}{\vec{z}^{\,2}}\right)+\frac{3}{2}\ln\frac{\vec{x}^{\,2}}{\vec{z}^{\,2}}\left(\frac{1}{\vec{x}^{\,2}}-\frac{1}{\vec{z}^{\,2}}\right)\right], \qquad (96)$$

and $J_{\vec{z}}$ can be obtained from $J_{\vec{x}}$ via $\vec{x} \leftrightarrow \vec{z}$ substitution. These integrals can be calculated using their analytical properties. Let us consider J_1 and write it as

$$J_1(\vec{x},\vec{z}) = \frac{1}{\vec{z}^{\,2}} f(t)|_{t=-\frac{\vec{x}^{\,2}}{\vec{z}^{\,2}}}, \quad f(t) = \int\limits_0^1 du \int\limits_0^1 dv \frac{1}{u(1-u)-v(1-v)t}. \qquad (97)$$

Finding the imaginary part of $f(t+i0)$

$$\mathrm{Im}\, f(t+i0) = \pi \int\limits_0^1 du \int\limits_0^1 dv \delta(u(1-u)-v(1-v)t) = \frac{1}{\sqrt{t}} \ln\left|\frac{1+t}{1-t}\right| \theta(t) \qquad (98)$$

and restoring $f(t)$ as

$$f(t) = \frac{1}{\pi}\int\limits_0^\infty \frac{\mathrm{Im}\, f(t'+i0)}{t'-t} dt'. \qquad (99)$$

we arrive to Eq. (94). The other functions J were calculated in the similar way. Finally we get $f_2(\vec{x},\vec{z})$ in the form (37).

Although the whole integral in (90) converges, separate terms diverge, so that we use dimensional regularization to calculate them. We need the following integrals:

$$\int \frac{d^{2+2\epsilon}l}{\pi^{1+\epsilon}\Gamma(1-\epsilon)} \frac{1}{l^2(l+p)^2} = \frac{2\left(\ln\left(p^2\right)+\frac{1}{\epsilon}\right)}{p^2}, \qquad (100)$$

$$\int \frac{d^{2+2\epsilon}l}{\pi^{1+\epsilon}\Gamma(1-\epsilon)} \frac{\ln\left(l^2\right)}{(l+p)^2} = \frac{1}{2}\ln^2\left(p^2\right) + \frac{\ln\left(p^2\right)}{\epsilon} + \frac{1}{\epsilon^2} - \frac{\pi^2}{6}, \qquad (101)$$

$$\int \frac{d^{2+2\epsilon}l}{\pi^{1+\epsilon}\Gamma\left(1-\epsilon\right)}\frac{\ln\left(\frac{l^2}{\mu^2}\right)}{l^2(l+p)^2} =$$

$$-\frac{\frac{1}{2}\ln^2\left(p^2\right)+\frac{\ln\left(p^2\right)}{\epsilon}+\frac{1}{\epsilon^2}-\frac{\pi^2}{6}}{p^2}-\frac{2\left(\ln\left(p^2\right)+\frac{1}{\epsilon}\right)\ln\left(\frac{\mu^2}{p^2}\right)}{p^2}, \tag{102}$$

$$\int \frac{d^{2+2\epsilon}l}{\pi^{1+\epsilon}\Gamma\left(1-\epsilon\right)}\frac{\ln\left(\frac{l^2}{\mu^2}\right)}{(l+p)^2(l+\mu)^2} = \frac{\left(\ln(p-\mu)^2+\frac{1}{2}\ln\left(\frac{p^2}{\mu^2}\right)+\frac{1}{\epsilon}\right)\ln\left(\frac{p^2}{\mu^2}\right)}{(p-\mu)^2}, \tag{103}$$

$$\int \frac{d^{2+2\epsilon}l}{\pi^{1+\epsilon}\Gamma\left(1-\epsilon\right)}\frac{\ln\left(\frac{l^2}{\mu^2}\right)}{l^2(l+p)^2(l-q)^2} =$$

$$= \frac{-\frac{1}{2}\ln^2\left(p^2\right)-\frac{\ln\left(p^2\right)}{\epsilon}-\frac{1}{\epsilon^2}+\frac{\pi^2}{6}}{p^2q^2}+\frac{\left(\ln\left(q^2\right)+\frac{1}{\epsilon}\right)\ln\left(\frac{q^2}{\mu^2}\right)}{q^2(p+q)^2}$$

$$-\frac{\ln\left(\frac{(p+q)^2}{q^2}\right)\ln\left(\frac{p^2q^2}{\mu^4}\right)\left((p+q)^2-q^2\right)}{2p^2q^2(p+q)^2}-\frac{2I\left(p^2,q^2,(p+q)^2\right)\left(p^2q^2-(p\,q)^2\right)}{p^2q^2(p+q)^2}$$

$$+\frac{\left(\ln\left(p^2\right)+\frac{1}{\epsilon}\right)\ln\left(\frac{p^2}{\mu^2}\right)}{p^2}\left(\frac{1}{q^2}+\frac{1}{(p+q)^2}\right)+\frac{\ln\left(\frac{(p+q)^2}{p^2}\right)\ln\left(\frac{p^2q^2}{\mu^4}\right)}{2q^2(p+q)^2}, \tag{104}$$

$$\int \frac{d^{2+2\epsilon}l}{\pi^{1+\epsilon}\Gamma\left(1-\epsilon\right)}\frac{\ln\left(\frac{l^2}{\mu^2}\right)(l+q)^2}{(l+p)^2(l+\mu)^2} = \frac{1}{2}\ln^2\left(\mu^2\right)+\frac{\ln\left(\mu^2\right)}{\epsilon}+\frac{1}{\epsilon^2}-\frac{\pi^2}{6}$$

$$+\frac{(q-p)(p-\mu)\ln^2\left(\frac{p^2}{\mu^2}\right)}{2(p-\mu)^2}-\frac{\ln\left(\frac{p^2\mu^2}{(p-\mu)^4}\right)(q-p)(p-\mu)\ln\left(\frac{p^2}{\mu^2}\right)}{2(p-\mu)^2}$$

$$+\frac{\left(\ln(p-\mu)^2+\frac{1}{2}\ln\left(\frac{p^2}{\mu^2}\right)+\frac{1}{\epsilon}\right)\ln\left(\frac{p^2}{\mu^2}\right)(q-p)^2}{(p-\mu)^2}$$

$$+2I\left(p^2,\mu^2,(p-\mu)^2\right)\left(\frac{p(q-p)\left((p\,\mu)-\mu^2\right)}{(p-\mu)^2}+\frac{\left((p\,\mu)-p^2\right)(q-p)\mu}{(p-\mu)^2}\right), \tag{105}$$

$$\int \frac{d^{2+2\epsilon}l}{\pi^{1+\epsilon}\Gamma\left(1-\epsilon\right)}\frac{\ln\left(\frac{l^2}{\mu^2}\right)(l+q)^2}{l^2(l+p)^2} = \frac{\left(\frac{1}{2}\ln^2\left(p^2\right)+\frac{\ln\left(p^2\right)}{\epsilon}+\frac{1}{\epsilon^2}-\frac{\pi^2}{6}\right)\left(p^2-q^2\right)}{p^2}$$

$$-\frac{\left(\ln\left(p^2\right)+\frac{1}{\epsilon}\right)\ln\left(\frac{\mu^2}{p^2}\right)\left(q^2-p^2+(p-q)^2\right)}{p^2}. \tag{106}$$

The function I, which appears in Eqs. (104) and (105), is given by

$$I(q^2,p^2,\mu^2)=\int\limits_0^1\frac{dx}{q^2(1-x)+p^2x-\mu^2x(1-x)}\ln\left(\frac{q^2(1-x)+p^2x}{\mu^2x(1-x)}\right)\,. \tag{107}$$

Using the integrals presented above we obtain

$$L\left(x,z\right)=\frac{x^2}{z^2}f_1\left(z,x\right)+\frac{x^2}{z^2}f_2\left(x,z\right)$$
$$+\ln\left(\frac{x^2}{z^2}\right)\ln\left(\frac{(x-z)^2}{z^2}\right)\left(\frac{1}{(x-z)^2}-\frac{x^2}{(x-z)^2z^2}\right)\,. \tag{108}$$

Adding the contributions of the functions $g^0(\vec{r}_1,\vec{r}_2;\vec{\rho})$ and $g(\vec{r}_1,\vec{r}_2;\vec{\rho})$ we arrive to Eq. (34).

Appendix B

Here we will describe the calculation of the commutators. The commutator necessary to eliminate the energy scale dependent terms in the difference of the forward kernels (39) is $\left[\hat{\mathcal{K}}^{(B)},\ln\left(\hat{\vec{q}}^{\,2}\right)\hat{\mathcal{K}}^{(B)}\right]$. We will calculate it in the momentum space via the identity

$$\langle\vec{q}|\left[\hat{\mathcal{K}}^{(B)},\,\ln\left(\hat{\vec{q}}^{\,2}\right)\hat{\mathcal{K}}^{(B)}\right]|\vec{q}\,'\rangle=\int d\vec{p}\langle\vec{q}|\left[\hat{\mathcal{K}}^{(B)},\,\ln\left(\hat{\vec{q}}^{\,2}\right)\right]|\vec{p}\rangle\langle\vec{p}|\hat{\mathcal{K}}^{(B)}|\vec{q}\,'\rangle. \tag{109}$$

Taking the LO forward kernel from Eq. (46) we get

$$\langle\vec{q}|\left[\hat{\mathcal{K}}^{(B)},\,\ln\left(\hat{\vec{q}}^{\,2}\right)\right]|\vec{p}\rangle=\frac{\alpha_sN_c}{\pi^2}\frac{\vec{p}^{\,2}}{(\vec{p}-\vec{q})^2\vec{q}^{\,2}}\ln\left(\frac{\vec{p}^{\,2}}{\vec{q}^{\,2}}\right). \tag{110}$$

Then for the whole commutator (109) in the momentum representation we have

$$\langle\vec{q}|\left[\hat{\mathcal{K}}^{(B)},\,\ln\left(\hat{\vec{q}}^{\,2}\right)\hat{\mathcal{K}}^{(B)}\right]|\vec{q}\,'\rangle=\frac{\alpha_s^2N_c^2}{2\pi^3}\frac{\vec{q}^{\,\prime 2}}{\vec{q}^{\,2}(\vec{q}-\vec{q}\,')^2}\ln\left(\frac{(\vec{q}-\vec{q}\,')^4}{\vec{q}^{\,2}\vec{q}^{\,\prime 2}}\right)\ln\left(\frac{\vec{q}^{\,\prime 2}}{\vec{q}^{\,2}}\right). \tag{111}$$

Now we will rewrite this result in the coordinate space. Since we need the operator in the Möbuis representation, i.e. with the matrix element equal to 0 at $\vec{r}=0$, we should Fourier transform this expression and subtract from it its value at $\vec{r}=0$. This subtraction allows us to cancel the singularity at $\vec{q}=0$ in Eq. (111). To find the Fourier transform it is convenient to rewrite Eq. (111) as

$$\langle\vec{q}|\left[\hat{\mathcal{K}}^{(B)},\,\ln\left(\hat{\vec{q}}^{\,2}\right)\hat{\mathcal{K}}^{(B)}\right]|\vec{q}'\rangle=\frac{\alpha_s^2N_c^2}{2\pi^3}\left(\frac{1}{\vec{q}^{\,2}}+\frac{1}{(\vec{q}-\vec{q}^{\,\prime})^2}-\frac{2\vec{q}(\vec{q}-\vec{q}^{\,\prime})}{\vec{q}^{\,2}(\vec{q}-\vec{q}^{\,\prime})^2}\right)$$
$$\times\left(\ln^2\left(\frac{(\vec{q}-\vec{q}^{\,\prime})^2}{\vec{q}^{\,2}}\right)-\ln^2\left(\frac{(\vec{q}-\vec{q}^{\,\prime})^2}{\vec{q}^{\,\prime 2}}\right)\right) \tag{112}$$

and use the following integrals:

$$\int\frac{d\vec{q}}{2\pi}\int\frac{d\vec{k}}{2\pi}e^{i[\vec{q}\,\vec{r}+\vec{k}\,\vec{\rho}]}\frac{1}{\vec{q}^2}\ln^2\left(\frac{(\vec{q}+\vec{k})^2}{\vec{k}^{\,2}}\right)=\frac{1}{\vec{\rho}^2}\ln^2\left(\frac{(\vec{r}-\vec{\rho})^2}{\vec{r}^2}\right), \tag{113}$$

$$\int\frac{d\vec{q}}{2\pi}\int\frac{d\vec{k}}{2\pi}e^{i[\vec{q}\,\vec{r}+\vec{k}\,\vec{\rho}]}\frac{(\vec{q}\,\vec{k})}{\vec{q}^2\vec{k}^{\,2}}\ln^2\left(\frac{(\vec{k}+\vec{q})^2}{\vec{q}^2}\right)=-\frac{(\vec{r}\,\vec{\rho})}{\vec{r}^2\vec{\rho}^2}\ln^2\left(\frac{(\vec{\rho}-\vec{r})^2}{\vec{\rho}^2}\right), \tag{114}$$

$$\int\frac{d\vec{q}}{2\pi}\int\frac{d\vec{k}}{2\pi}e^{i[\vec{q}\,\vec{r}+\vec{k}\,\vec{\rho}]}\frac{(\vec{q}\,\vec{k})}{\vec{q}^2\vec{k}^{\,2}}\ln^2\left(\frac{\vec{k}^{\,2}}{\vec{q}^2}\right)=-\frac{(\vec{r}\,\vec{\rho})}{\vec{r}^2\vec{\rho}^2}\ln^2\left(\frac{\vec{\rho}^2}{\vec{r}^2}\right), \tag{115}$$

$$\int\frac{d\vec{q}}{2\pi}\int\frac{d\vec{k}}{2\pi}\frac{1}{\vec{k}^{\,2}}\ln^2\left(\frac{\vec{k}^{\,2}}{\vec{q}^2}\right)\left(e^{i[\vec{q}\,\vec{r}+\vec{k}\,\vec{\rho}]}-e^{i[\vec{q}\,\vec{r}+\vec{k}\,(\vec{\rho}-\vec{r})]}\right)$$
$$=\frac{1}{\vec{r}^2}\left(\ln^2\left(\frac{\vec{\rho}^2}{\vec{r}^2}\right)-\ln^2\left(\frac{(\vec{\rho}-\vec{r})^2}{\vec{r}^2}\right)\right), \tag{116}$$

$$\int\frac{d\vec{q}}{2\pi}\int\frac{d\vec{k}}{2\pi}\frac{1}{\vec{q}^2}\ln^2\left(\frac{\vec{k}^{\,2}}{\vec{q}^2}\right)\left(e^{i[\vec{q}\,(\vec{r}-\vec{\rho})-\vec{k}\,\vec{\rho}]}-e^{-i[\vec{q}+\vec{k}]\,\vec{\rho}}\right)=\frac{1}{\vec{\rho}^2}\ln^2\left(\frac{(\vec{\rho}-\vec{r})^2}{\vec{\rho}^2}\right). \tag{117}$$

As the result, we obtain

$$\langle\vec{r}|\left[\hat{\mathcal{K}}^{(B)},\,\ln\left(\hat{\vec{q}}^{\,2}\right)\hat{\mathcal{K}}^{(B)}\right]_M|\vec{r}'\rangle$$
$$=\int\frac{d\vec{q}}{2\pi}\frac{d\vec{q}^{\,\prime}}{2\pi}\langle\vec{q}|\left[\hat{\mathcal{K}}^{(B)},\,\ln\left(\hat{\vec{q}}^{\,2}\right)\hat{\mathcal{K}}^{(B)}\right]|\vec{q}'\rangle\left(e^{i\,\vec{q}\,\vec{r}-i\,\vec{q}^{\,\prime}\vec{r}^{\,\prime}}-e^{-i\,\vec{q}^{\,\prime}\vec{r}^{\,\prime}}\right)$$
$$=-\frac{\alpha_s^2N_c^2}{2\pi^3}\frac{\vec{r}^{\,2}}{(\vec{r}-\vec{r}')^2\vec{r}^{\,\prime 2}}\ln\left(\frac{\vec{r}^{\,2}}{\vec{r}^{\,\prime 2}}\right)\ln\left(\frac{\vec{r}^{\,2}\vec{r}^{\,\prime 2}}{(\vec{r}-\vec{r}')^4}\right), \tag{118}$$

which eliminates a part of the difference between the kernels in Eq. (39).

A natural generalization of the previous commutator to the nonforward case is

$$\left[\hat{\mathcal{K}}^{(B)},\, \ln\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right)\hat{\mathcal{K}}^{(B)}\right]. \tag{119}$$

We will also calculate it in the momentum representation and then Fourier transform it to the coordinate space. In the momentum representation we have

$$\langle\vec{q}_1,\vec{q}_2|\left[\hat{\mathcal{K}}^{(B)},\, \ln\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right)\hat{\mathcal{K}}^{(B)}\right]|\vec{q}_1^{\,\prime},\vec{q}_2^{\,\prime}\,\rangle$$

$$= \int d\vec{p}_1 d\vec{p}_2 \langle\vec{q}_1,\vec{q}_2|\left[\hat{\mathcal{K}}^{(B)}\,\ln\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right)\right]|\vec{p}_1,\vec{p}_2\rangle\langle\vec{p}_1,\vec{p}_2|\hat{\mathcal{K}}^{(B)}|\vec{q}_1^{\,\prime},\vec{q}_2^{\,\prime}\,\rangle$$

$$= \delta\left(\vec{q}-\vec{q}^{\,\prime}\right)\int d\vec{k}_1 \frac{\mathcal{K}_r^{(B)}(\vec{q}_1,\vec{q}_1-\vec{k}_1,\vec{q})\mathcal{K}_r^{(B)}(\vec{q}_1-\vec{k}_1,\vec{q}_1^{\,\prime},\vec{q}^{\,\prime})}{\vec{q}_1^{\,2}\vec{q}_2^{\,2}(\vec{q}_1-\vec{k}_1)^{\,2}(\vec{q}_2+\vec{k}_1)^{\,2}}$$

$$\times\ln\left(\frac{(\vec{q}_1-\vec{k}_1)^{\,2}(\vec{q}_2+\vec{k}_1)^{\,2}}{\vec{q}_1^{\,2}\vec{q}_2^{\,2}}\right)$$

$$+\delta\left(\vec{q}-\vec{q}^{\,\prime}\right)\frac{\mathcal{K}_r^{(B)}(\vec{q}_1,\vec{q}_1^{\,\prime},\vec{q})}{\vec{q}_1^{\,2}\vec{q}_2^{\,2}}\left(\omega\left(\vec{q}_1^{\,\prime 2}\right)+\omega\left(\vec{q}_2^{\,\prime 2}\right)\right)\ln\left(\frac{\vec{q}_1^{\,\prime 2}\vec{q}_2^{\,\prime 2}}{\vec{q}_1^{\,2}\vec{q}_2^{\,2}}\right). \tag{120}$$

It is more convenient for the integration to rewrite the commutator in the following form

$$\left(\frac{\pi^{\,2}}{\alpha N}\right)^{2}\langle\vec{r}_1,\vec{r}_2|\left[\hat{\mathcal{K}}^{(B)},\ln\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right)\hat{\mathcal{K}}^{(B)}\right]|\vec{r}_1^{\,\prime},\vec{r}_2^{\,\prime}\rangle$$

$$= \int\frac{d\vec{q}_1}{2\pi}\frac{d\vec{q}_2}{2\pi}\frac{d\vec{k}_1}{2\pi}\frac{d\vec{k}_2}{2\pi}e^{i[\vec{q}_1\,\vec{r}_{11'}+\vec{q}_2\,\vec{r}_{22'}+(\vec{k}_1+\vec{k}_2)\,\vec{r}_{1'2'}]}$$

$$\ln\left(\frac{(\vec{q}_1-\vec{k}_1)^{\,2}(\vec{q}_2+\vec{k}_1)^{\,2}}{\vec{q}_1^{\,2}\vec{q}_2^{\,2}}\right)\left[\frac{1}{\vec{k}_2^{\,2}}\left(\frac{1}{\vec{k}_1^{\,2}}+\frac{\vec{k}_1}{\vec{k}_1^{\,2}}\left(\frac{\vec{q}_2}{\vec{q}_2^{\,2}}-\frac{\vec{q}_1}{\vec{q}_1^{\,2}}\right)-\frac{(\vec{q}_1\,\vec{q}_2)}{\vec{q}_1^{\,2}\vec{q}_2^{\,2}}\right)\right.$$

$$+\frac{1}{\vec{k}_1^{\,2}}\left(\frac{\vec{k}_2}{\vec{k}_2^{\,2}}\left(\frac{\vec{q}_2+\vec{k}_1}{(\vec{q}_2+\vec{k}_1)^{\,2}}-\frac{\vec{q}_1-\vec{k}_1}{(q_1-\vec{k}_1)^{\,2}}\right)-\frac{(\vec{q}_1-\vec{k}_1)(\vec{q}_2+\vec{k}_1)}{(\vec{q}_1-\vec{k}_1)^{\,2}(\vec{q}_2+\vec{k}_1)^{\,2}}\right)$$

$$+\left(\frac{\vec{k}_1}{\vec{k}_1^{\,2}}\left(\frac{\vec{q}_2}{\vec{q}_2^{\,2}}-\frac{\vec{q}_1}{\vec{q}_1^{\,2}}\right)\right)$$

$$\times\left(\frac{\vec{k}_2}{\vec{k}_2^{\,2}}\left(\frac{\vec{q}_2+\vec{k}_1}{(\vec{q}_2+\vec{k}_1)^{\,2}}-\frac{\vec{q}_1-\vec{k}_1}{(\vec{q}_1-\vec{k}_1)^{\,2}}\right)-\frac{(\vec{q}_1-\vec{k}_1)(\vec{q}_2+\vec{k}_1)}{(\vec{q}_1-\vec{k}_1)^{\,2}(\vec{q}_2+\vec{k}_1)^{\,2}}\right)$$

$$
-\frac{(\vec{q}_1\,\vec{q}_2)}{\vec{q}_1^{\,2}\vec{q}_2^{\,2}}\left(\frac{\vec{k}_2}{\vec{k}_2^{\,2}}\left(\frac{\vec{q}_2+\vec{k}_1}{(\vec{q}_2+\vec{k}_1)^{\,2}}-\frac{\vec{q}_1-\vec{k}_1}{(\vec{q}_1-\vec{k}_1)^{\,2}}\right)-\frac{(\vec{q}_1-\vec{k}_1)(\vec{q}_2+\vec{k}_1)}{(\vec{q}_1-\vec{k}_1)^{\,2}(\vec{q}_2+\vec{k}_1)^{\,2}}\right)\Bigg]
$$

$$
-\frac{1}{4}\int\frac{d\vec{q}_1}{2\pi}\frac{d\vec{q}_2}{2\pi}\frac{d\vec{k}_1}{2\pi}\frac{d\vec{k}_2}{2\pi}e^{i[\vec{q}_1\,\vec{r}_{11'}+\vec{q}_2\,\vec{r}_{22'}+\vec{k}_1\,\vec{r}_{1'2'}]}\ln\left(\frac{(\vec{q}_1-\vec{k}_1)^{\,2}(\vec{q}_2+\vec{k}_1)^{\,2}}{\vec{q}_1^{\,2}\vec{q}_2^{\,2}}\right)
$$

$$
\times\left(\frac{2}{\vec{k}_2^{\,2}}+\frac{2\vec{k}_2}{\vec{k}_2^{\,2}}\left(\frac{\vec{q}_2+\vec{k}_1-\vec{k}_2}{(\vec{q}_2+\vec{k}_1-\vec{k}_2)^{\,2}}+\frac{\vec{q}_1-\vec{k}_1-\vec{k}_2}{(\vec{q}_1-\vec{k}_1-\vec{k}_2)^{\,2}}\right)\right.
$$

$$
\left.+\frac{1}{(\vec{q}_1-\vec{k}_1-\vec{k}_2)^{\,2}}+\frac{1}{(\vec{q}_1-\vec{k}_1-\vec{k}_2)^{\,2}}\right)
$$

$$
\times\left(\frac{1}{\vec{k}_1^{\,2}}+\frac{\vec{k}_1}{\vec{k}_1^{\,2}}\left(\frac{\vec{q}_2}{\vec{q}_2^{\,2}}-\frac{\vec{q}_1}{\vec{q}_1^{\,2}}\right)-\frac{(\vec{q}_1\,\vec{q}_2)}{\vec{q}_1^{\,2}\vec{q}_2^{\,2}}\right). \qquad (121)
$$

This expression can be straightforwardly Fourier transformed with the help of the integrals presented in Ref. [17]. Finally we obtain

$$
4\left(\frac{\pi^{\,2}}{\alpha_s N_c}\right)^2\langle\vec{r}_1,\vec{r}_2|\left[\hat{\mathcal{K}}^{(B)},\ln\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right)\hat{\mathcal{K}}^{(B)}\right]|\vec{r}_1^{\,\prime},\vec{r}_2^{\,\prime}\rangle
$$

$$
=4\left(\frac{\pi^{\,2}}{\alpha_s N_c}\right)^2\langle\vec{r}_1,\vec{r}_2|\left[\hat{\mathcal{K}}^{(B)},\ln\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right)\hat{\mathcal{K}}^{(B)}\right]_M|\vec{r}_1^{\,\prime},\vec{r}_2^{\,\prime}\rangle
$$

$$
+\left(-\frac{2\left(\vec{r}_{11'}\vec{r}_{12'}\right)\ln\left(\frac{\vec{r}_{11'}^{\,2}\vec{r}_{12'}^{\,2}}{\vec{r}_{1'2'}^{\,4}}\right)}{\vec{r}_{11'}^{\,2}\vec{r}_{12'}^{\,2}\vec{r}_{1'2'}^{\,2}}+\delta\left(\vec{r}_{1'2'}\right)(\dots)+(1\leftrightarrow 2,1'\leftrightarrow 2')\right), \qquad (122)
$$

where

$$
4\left(\frac{\pi^{\,2}}{\alpha_s N_c}\right)^2\langle\vec{r}_1,\vec{r}_2|\left[\hat{\mathcal{K}}^{(B)},\ln\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right)\hat{\mathcal{K}}^{(B)}\right]_M|\vec{r}_1^{\,\prime},\vec{r}_2^{\,\prime}\rangle=2\pi\delta\left(\vec{r}_{11'}\right)
$$

$$
\times\left\{2I\left(\vec{r}_{12}^{\,2},\vec{r}_{22'}^{\,2},\vec{r}_{12'}^{\,2}\right)\left(\frac{\left(\vec{r}_{22'}\vec{r}_{12'}\right)^2}{\vec{r}_{22'}^{\,2}\vec{r}_{12'}^{\,2}}-1\right)\right.
$$

$$
\left.-\frac{\left(\vec{r}_{22'}\vec{r}_{12'}\right)}{2\vec{r}_{22'}^{\,2}\vec{r}_{12'}^{\,2}}\ln^2\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{12'}^{\,2}}+\left(\frac{\left(\vec{r}_{22'}\vec{r}_{12'}\right)}{\vec{r}_{22'}^{\,2}\vec{r}_{12'}^{\,2}}-\frac{1}{2\vec{r}_{12'}^{\,2}}\right)\ln\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{12'}^{\,2}}\ln\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{22'}^{\,2}}\right\}
$$

$$
-2\int d\vec{\rho}\left(\frac{\vec{r}_{1'2'}^{\,2}}{\vec{r}_{1'\rho}^{\,2}\vec{r}_{2'\rho}^{\,2}}\delta\left(\vec{r}_{22'}\right)\frac{\left(\vec{r}_{12}\vec{r}_{1'2}\right)}{\vec{r}_{11'}^{\,2}\vec{r}_{1'2}^{\,2}}\ln\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{1'2}^{\,2}}-\frac{\delta\left(\vec{r}_{2\rho}\right)}{\vec{r}_{2'\rho}^{\,2}}\frac{\left(\vec{r}_{1\rho}\vec{r}_{1'\rho}\right)}{\vec{r}_{11'}^{\,2}\vec{r}_{1'\rho}^{\,2}}\ln\frac{\vec{r}_{1\rho}^{\,2}}{\vec{r}_{1'\rho}^{\,2}}\right.
$$

$$-\frac{\delta\left(\vec{r}_{22'}\right)}{\vec{r}^{\,2}_{1'\rho}}\frac{\left(\vec{r}_{12}\vec{r}_{\rho 2}\right)}{\vec{r}^{\,2}_{1\rho}\vec{r}^{\,2}_{\rho 2}}\ln\frac{\vec{r}^{\,2}_{12}}{\vec{r}^{\,2}_{\rho 2}}\Bigg)$$

$$+\vec{r}^{\,2}_{12}\left(\frac{\ln\left(\frac{\vec{r}^{\,4}_{12}}{\vec{r}^{\,2}_{12'}\vec{r}^{\,2}_{21'}}\right)}{2\vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{22'}\vec{r}^{\,2}_{1'2'}}+\frac{\ln\left(\frac{\vec{r}^{\,4}_{1'2'}}{\vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{22'}}\right)}{\vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{12'}\vec{r}^{\,2}_{22'}}+\frac{\ln\left(\frac{\vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{21'}\vec{r}^{\,2}_{22'}}{\vec{r}^{\,2}_{12}\vec{r}^{\,4}_{1'2'}}\right)}{\vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{21'}\vec{r}^{\,2}_{1'2'}}\right.$$

$$+\frac{\ln\left(\frac{\vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{21'}\vec{r}^{\,2}_{22'}}{\vec{r}^{\,2}_{12'}\vec{r}^{\,4}_{1'2'}}\right)}{\vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{1'2'}}\left(\frac{\vec{r}^{\,2}_{12'}}{\vec{r}^{\,2}_{22'}}-1\right)$$

$$\left.+\frac{\ln\left(\frac{\vec{r}^{\,2}_{12'}\vec{r}^{\,2}_{21'}}{\vec{r}^{\,4}_{12}}\right)}{2\vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{22'}}+\frac{\ln\left(\frac{\vec{r}^{\,2}_{12}}{\vec{r}^{\,2}_{12'}}\right)}{\vec{r}^{\,2}_{11'}\vec{r}^{\,2}_{12'}}\right)+\left(1\leftrightarrow 2, 1'\leftrightarrow 2'\right). \tag{123}$$

One can check that this expression vanishes as $\vec{r}_2 \to \vec{r}_1$ and hence has the Möbius property. Next, one needs the integrals from appendix A to calculate its forward form and to see that it exactly coincides with the result (118). Unfortunately it is clear that this commutator does not eliminate the discrepancy between the kernels.

Another generalization of $\left[\hat{\mathcal{K}}^{(B)}, \ln\left(\hat{\vec{q}}^{\,2}\right)\hat{\mathcal{K}}^{(B)}\right]$ to the nonforward case is $\left[\hat{\mathcal{K}}^{(B)}, \hat{\mathcal{K}}^{(B)}\ln\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right)\right]$. We will calculate its matrix element in the coordinate space via the identity

$$\langle\vec{r}_1,\vec{r}_2|\left[\hat{\mathcal{K}}^{(B)},\ \hat{\mathcal{K}}^{(B)}\ln\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right)\right]|\vec{r}_1^{\,\prime},\vec{r}_2^{\,\prime}\rangle$$

$$=\int d\vec{\rho}_1 d\vec{\rho}_2\langle\vec{r}_1,\vec{r}_2|\hat{\mathcal{K}}^{(B)}|\vec{\rho}_1,\rho_2\rangle\langle\vec{\rho}_1,\vec{\rho}_2|\left[\hat{\mathcal{K}}^{(B)},\ \ln\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right)\right]|\vec{r}_1^{\,\prime},\vec{r}_2^{\,\prime}\rangle. \tag{124}$$

Taking Fourier transform we get

$$\langle\vec{r}_1,\vec{r}_2|\ln\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right)|\vec{r}_1^{\,\prime},\vec{r}_2^{\,\prime}\rangle=-\frac{2}{2\pi}\left(\frac{\delta\left(\vec{r}_{11'}\right)}{\vec{r}^{\,2}_{22'}}+\frac{\delta\left(\vec{r}_{22'}\right)}{\vec{r}^{\,2}_{11'}}\right). \tag{125}$$

However, this operator is not Möbius since its matrix element does not vanish as $\vec{r}_1 \to \vec{r}_2$. Yet we can change the matrix element adding some terms independent of $\vec{r}_1$ or of $\vec{r}_2$ so that it satisfies the Möbius property. We have

$$\langle\vec{r}_1,\vec{r}_2|\ln\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right)_M|\vec{r}_1^{\,\prime},\vec{r}_2^{\,\prime}\rangle=-\frac{2}{2\pi}\left(\frac{\delta\left(\vec{r}_{11'}\right)}{\vec{r}^{\,2}_{22'}}-\frac{\delta\left(\vec{r}_{11'}\right)}{\vec{r}^{\,2}_{12'}}\right)+\left(1\leftrightarrow 2, 1'\leftrightarrow 2'\right). \tag{126}$$

Thus we arrive to

$$\frac{2\pi^3}{\alpha_s N_c}\langle \vec{r}_1, \vec{r}_2| \left[\hat{\mathcal{K}}^{(B)},\, \ln\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right)_M\right] |\vec{r}_1^{\,\prime}, \vec{r}_2^{\,\prime}\, \rangle = 2\pi\delta\left(\vec{r}_{11'}\right)\frac{\left(\vec{r}_{12}, \vec{r}_{12'}\right)}{\vec{r}_{12'}^{\,2}\vec{r}_{22'}^{\,2}}\ln\left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{12'}^{\,2}}\right)$$

$$+\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{11'}^{\,2}\vec{r}_{21'}^{\,2}}\left(\frac{1}{\vec{r}_{1'2'}^{\,2}} - \frac{1}{\vec{r}_{22'}^{\,2}}\right) + \frac{1}{\vec{r}_{11'}^{\,2}\vec{r}_{1'2'}^{\,2}}\left(\frac{\vec{r}_{12'}^{\,2}}{\vec{r}_{22'}^{\,2}} - 1\right) + (1 \leftrightarrow 2, 1' \leftrightarrow 2'). \quad (127)$$

This matrix element tends to zero as $\vec{r}_1 \to \vec{r}_2$ and hence can be convolved with the kernel. Finally we get

$$\left(\frac{2\pi^2}{\alpha_s N_c}\right)^2 \langle \vec{r}_1, \vec{r}_2| \left[\hat{\mathcal{K}}^{(B)},\, \hat{\mathcal{K}}^{(B)}\ln\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right)_M\right] |\vec{r}_1^{\,\prime}, \vec{r}_2^{\,\prime}\, \rangle$$

$$= 2\int d\vec{\rho}\,\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{1'2'}^{\,2}\vec{r}_{1\rho}^{\,2}\vec{r}_{2\rho}^{\,2}}\left(\left(\delta\left(\vec{r}_{11'}\right) - \delta\left(\vec{r}_{1'\rho}\right)\right)\frac{\left(\vec{r}_{21'}\,\vec{r}_{1'2'}\right)}{\vec{r}_{22'}^{\,2}}\ln\left(\frac{\vec{r}_{21'}^{\,2}}{\vec{r}_{1'2'}^{\,2}}\right)\right.$$

$$\left. +\delta\left(\vec{r}_{11'}\right)\frac{\left(\vec{r}_{1'2'}\,\vec{r}_{1'\rho}\right)}{\vec{r}_{2'\rho}^{\,2}}\ln\left(\frac{\vec{r}_{1'\rho}^{\,2}}{\vec{r}_{1'2'}^{\,2}}\right)\right)$$

$$+\vec{r}_{12}^{\,2}\left(\frac{\ln\left(\frac{\vec{r}_{12'}^{\,2}\vec{r}_{21'}^{\,2}}{\vec{r}_{12}^{\,2}\vec{r}_{1'2'}^{\,2}}\right)}{\vec{r}_{11'}^{\,2}\vec{r}_{22'}^{\,2}\vec{r}_{1'2'}^{\,2}} + \frac{\ln\left(\frac{\vec{r}_{11'}^{\,2}\vec{r}_{22'}^{\,2}}{\vec{r}_{12}^{\,2}\vec{r}_{1'2'}^{\,2}}\right)}{\vec{r}_{11'}^{\,2}\vec{r}_{21'}^{\,2}}\left(\frac{1}{\vec{r}_{1'2'}^{\,2}} - \frac{1}{\vec{r}_{22'}^{\,2}}\right)\right)$$

$$+\frac{\ln\left(\frac{\vec{r}_{11'}^{\,2}\vec{r}_{22'}^{\,2}}{\vec{r}_{12'}^{\,2}\vec{r}_{21'}^{\,2}}\right)}{\vec{r}_{11'}^{\,2}\vec{r}_{1'2'}^{\,2}}\left(\frac{\vec{r}_{12'}^{\,2}}{\vec{r}_{22'}^{\,2}} - 1\right) + (1 \leftrightarrow 2, 1' \leftrightarrow 2'). \quad (128)$$

This matrix element has the Mobius property since it vanishes at $\vec{r}_1 = \vec{r}_2$.

For the forward case we have

$$\left(\frac{2\pi^2}{\alpha_s N_c}\right)^2 \int d\vec{r}_{1'}d\vec{r}_{2'}\langle \vec{r}_1, \vec{r}_2| \left[\hat{\mathcal{K}}^{(B)},\, \hat{\mathcal{K}}^{(B)}\ln\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right)_M\right] |\vec{r}_1^{\,\prime}, \vec{r}_2^{\,\prime}\, \rangle\delta(\vec{r}_{1'2'} - \vec{r}^{\,\prime})$$

$$= -4\pi\frac{\vec{r}^{\,2}}{(\vec{r} - \vec{r}')^2\vec{r}^{\,\prime 2}}\ln\left(\frac{\vec{r}^{\,2}}{\vec{r}^{\,\prime 2}}\right)\ln\left(\frac{\vec{r}^{\,2}\vec{r}^{\,\prime 2}}{(\vec{r} - \vec{r}')^4}\right). \quad (129)$$

Here $\vec{r} = \vec{r}_{12}$ and we used the integrals from appendix A to reproduce Eq. (111).

Appendix C

The integrals (48) and (52) were calculated performing firstly the angular integration as well as in Ref. [18], with the use of the expansion over the Chebyshev polinomials

$$\frac{1-t^2}{1-2tx+t^2} = 1+2\sum_{n=1}^{\infty} t^n T_n\left(x\right), \; \ln\left(1-2tx+t^2\right) = -2\sum_{n=1}^{\infty} \frac{t^n}{n} T_n\left(x\right), \; |t|<1, \tag{130}$$

and the relations

$$2T_n(x)T_m(x) = T_{n+m}(x) + T_{n-m}(x), \;\; T_0(x) = 1, \qquad T_{-n}(x) = T_n(x),$$

$$\int_{-\pi}^{\pi} \frac{d\phi}{\pi} e^{in\phi} T_m\left(\cos\phi\right) = 2\int_{0}^{\pi} \frac{d\phi}{\pi} e^{in\phi} T_m\left(\cos\phi\right) = \delta_{nm}\left(1+\delta_{n0}\right) \; . \tag{131}$$

After this the integral (48) can be written as $-\partial/(\partial\gamma)J_1(n,\gamma)$, where

$$J_1(n,\gamma) = \int_0^1 \frac{dt}{1-t}\left[t^{\gamma+\frac{n}{2}-1}\ln t - 2\left(t^{\gamma+\frac{n}{2}-1} + t^{\gamma-\frac{n}{2}-1} - 2\right)\ln(1-t)\right.$$

$$\left. -2t^{\gamma-1}\sum_{l=1}^{n-1} t^{l-\frac{n}{2}} \frac{\left(1-t^{n-l}\right)}{l}\right] + (\gamma \leftrightarrow (1-\gamma)). \tag{132}$$

Here and below we assume that $n = |n|$. The integral (132) is taken using the relations

$$\int_0^1 \frac{dt}{1-t}(t^{a-1}-1) = \psi(1) - \psi(a),$$

$$2\int_0^1 \frac{dt}{1-t}(t^{a-1}-1)\ln(1-t) = \psi'(1) - \psi'(a) + \left(\psi(1)-\psi(a)\right)^2,$$

$$-2\int_0^1 \frac{dt}{1-t} t^{\gamma-1}\sum_{l=1}^{n} t^{l-\frac{n}{2}} \frac{\left(1-t^{n-l}\right)}{l} + (\gamma \leftrightarrow (1-\gamma))$$

$$= -2\sum_{l=1}^{n-1}\sum_{m=1}^{n-l} \frac{1}{l}\left(\frac{1}{\gamma-\frac{n}{2}+(l+m-1)} + \frac{1}{-\gamma+\frac{n}{2}+(l+m-n)}\right)$$

$$= 2\sum_{l=1}^{n-1}\sum_{k=l}^{n-1}\left(\frac{1}{\gamma-\frac{n}{2}+k}\frac{1}{\gamma-\frac{n}{2}+k-l}\right) = \sigma_1^2(\gamma,n)-\sigma_2(\gamma,n)\ , \qquad (133)$$

where

$$\sigma_1(\gamma,n) = \sum_{m=1}^{n}\frac{1}{\gamma-\frac{n}{2}-1+m},\ \ \sigma_2(\gamma,n) = \sum_{m=1}^{n}\frac{1}{(\gamma-\frac{n}{2}-1+m)^2},$$

$$\sigma_1(\gamma,n) = -\sigma_1(1-\gamma,n),\ \ \sigma_2(\gamma,n) = \sigma_2(1-\gamma,n). \qquad (134)$$

After this one can exclude $\psi(a-\frac{n}{2})$ and $\psi'(a-\frac{n}{2})$ exploiting the properties

$$\psi(a-\frac{n}{2}) = \psi(a+\frac{n}{2})-\sigma_1(a,n),\ \ \psi'(a-\frac{n}{2}) = \psi'(a+\frac{n}{2})+\sigma_2(a,n). \quad (135)$$

Finally, using the relation

$$\psi'(a+\frac{n}{2})+\psi'(1-a+\frac{n}{2}) =$$

$$-\sigma_2(a,n)+6\psi'(1)+\Big(\psi(1-a+\frac{n}{2})-\psi(a+\frac{n}{2})+\sigma_1(a,n)\Big)^2 \qquad (136)$$

one obtains

$$J_1(n,\gamma) = 2\psi'(1)-\chi^2(n,\gamma),\ \ -\frac{\partial}{\partial\gamma}J_1(n,\gamma) = 2\chi'(n,\gamma)\chi(n,\gamma), \qquad (137)$$

that gives Eq. (48). Let us add for completeness that Eqs. (135) and (136) follow from the properties

$$\psi(x+1) = \frac{1}{x}+\psi(x),\ \ \psi'(x+1) = -\frac{1}{x^2}+\psi'(x),$$

$$\psi'(x)+\psi'(1-x) = 6\psi'(1)+(\psi(x)-\psi(1-x))^2. \qquad (138)$$

In turn, these properties follow from the definition $\psi(x) = (\ln\Gamma(x))'$ and the properties

$$\Gamma(1+x) = x\Gamma(x),\ \ \Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin\pi x}. \qquad (139)$$

The integral (52) is calculated in the same way. We present here the results of the integration of separate terms. Most of necessary integrals can be found in [18] and [12]. The integral

$$J_2(n,\gamma) = \int\frac{d\vec{r}\,'}{\pi}\left(\frac{1}{(\vec{r}-\vec{r}\,')^2}-\frac{1}{\vec{r}\,'^2}\right)$$

$$\times \ln\left(\frac{(\vec{r}-\vec{r}\,')^2}{\vec{r}\,'^2}\right)\left(2\left(\frac{\vec{r}\,'^2}{\vec{r}^{\,2}}\right)^{\gamma} e^{in(\phi_{\vec{r}\,'}-\phi_{\vec{r}})}-1\right)$$

$$= \chi^2(n,\gamma) - \chi'(n,\gamma) - \frac{4\gamma\chi(n,\gamma)}{\gamma^2-\frac{n^2}{4}} \tag{140}$$

is calculated quite analogously to $J_1(n,\gamma)$. The calculation of the integral

$$\int d\vec{r}\,' e^{in(\phi_{\vec{r}\,'}-\phi_{\vec{r}})}\left(\frac{\vec{r}\,'^2}{\vec{r}^{\,2}}\right)^{\gamma-1}\frac{f_1\left(\vec{r},\vec{r}\,'\right)}{2\pi} = -\Phi\left(n,\gamma\right) - \Phi\left(n,1-\gamma\right), \tag{141}$$

where $f_1\left(\vec{r},\vec{r}\,'\right)$ is defined in Eq. (36) and $\Phi\left(n,\gamma\right)$ in Eq. (54), does not meet difficulties after the decomposition

$$\frac{1}{\left(\vec{x}-\vec{y}\right)^2\left(\vec{x}+\vec{y}\right)^2} = \frac{1}{2\left(\vec{x}^{\,2}+\vec{y}^{\,2}\right)}\left(\frac{1}{\left(\vec{x}-\vec{y}\right)^2}+\frac{1}{\left(\vec{x}+\vec{y}\right)^2}\right).$$

Finally, the integral

$$\int \frac{d\vec{r}\,'}{\pi} e^{in(\phi_{\vec{r}\,'}-\phi_{\vec{r}})}\left(\frac{\vec{r}\,'^2}{\vec{r}^{\,2}}\right)^{\gamma-1} f_2\left(\vec{r},\vec{r}\,'\right) = F(n,\gamma)\ , \tag{142}$$

where $f_2\left(\vec{r},\vec{r}\,'\right)$ is defined in Eq. (37) and $F\left(n,\gamma\right)$ in Eq. (55), can be performed using the relations

$$\int_0^\infty \frac{y^\alpha dy}{y+t^2} = -\frac{\pi t^{2\alpha}}{\sin\left(\pi\alpha\right)}, \quad \int_0^\infty dt\, t^\alpha \ln\left|\frac{1-t}{1+t}\right| = \frac{\pi\cos\left(\frac{\pi\alpha}{2}\right)}{(\alpha+1)\sin\left(\frac{\pi\alpha}{2}\right)} \tag{143}$$

after the angular integration.

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