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SURFACE IMPEDANCE OF SUPERCONDUCTORS IN WIDE FREQUENCY RANGES FOR WAKE FIELD CALCULATIONS

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Surface impedance of superconductors in wide frequency ranges for wake field calculations

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Abstract

An expression for surface impedance of superconductors suitable for numerical computations is given.

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1 Introduction

In considering problems of beam stability in accelerators it is important to know the wake fields generated by bunches of charged particles moving axially inside a metallic vacuum chamber. If the bunch is sufficiently short and is relativistic then the generated wake field contains modes of a wide frequency range. To calculate the fields it is necessary to know the surface impedance of the chamber metal for this large frequency range. The consideration of resistive wake fields of an axially moving relativistic line Gaussian bunch in a circular beam pipe at room temperature is based on the normal skin boundary conditions. If the beam pipe is at low (a few K^0) temperature, but in normal state, then the metal electron mean free path may exceed the skin depth and then anomalous skin or general Reuter-Sondheimer boundary conditions must be used. Such consideration was performed in [5].
The case of a superconducting beam pipe also deserves consideration. For fields with frequencies $h\omega > 2\epsilon_0$ (ϵ_0 is the energy gap of the superconductor) the surface impedance must go to that of the metal in norma state, but at $h\omega < 2\epsilon_0$ the the surface impedance is specific. The expressions for surface impedance in the base of BCS theory was given in [2] for type-1 superconductors. For the extreme anomalous limit approximation the expressions were significantly simplified.
The intention of the present paper is proceeding from general expressions of [2] to derive surface impedance expressions suitable for numerical computations.

2 Expression for the surface impedance of superconductor

To calculate the surface impedance $Z_s(\omega)$ of a metal in superconducting state we use the expressions derived in [1, 2] and used in [3]. Using the notations of this works we have for the case of specular electron reflection at the surface

$$Z_{ss}(\omega) \; = \; i \cdot \frac{8\omega}{c^2} \cdot \int\limits_0^{\infty} \frac{dq}{q^2 + K(q)} \tag{1s}$$

and for diffuse reflection case

$$Z_{sd}(\omega) \; = \; -i \cdot \frac{4\pi^2\omega}{c^2} \cdot \frac{1}{\int\limits_0^{\infty} \ln{(1 + K(q)/q^2)} dq} \tag{1d}$$

where

$$K(q) = -\frac{3}{c^2 h v_F \Lambda(0)} \cdot \int\limits_0^{\infty} dR \cdot \int\limits_{-1}^{1} e^{iqRu - R/l}(1-u^2) \cdot I(\omega, R, T) \cdot du \;, \tag{2}$$

$$I(\omega, R, T) = -\, i\pi \cdot \int\limits_{\epsilon_0 - h\omega}^{\epsilon_0} (1 - 2f(E + h\omega))$$

$$\times (g(E) \cdot \cos(\alpha\epsilon_2)) - i \sin(\alpha\epsilon_2)) e^{i\alpha\epsilon_1} \cdot dE$$

$$-i\pi \cdot \int\limits_{\epsilon_0}^{\infty} \Big[(1 - 2f(E + h\omega)) \cdot (g(E) \cdot \cos(\alpha\epsilon_2) - i \sin(\alpha\epsilon_2)) \cdot e^{i\alpha\epsilon_1}$$

$$-(1 - 2f(E)) \cdot (g(E) \cdot \cos(\alpha\epsilon_1) + i \sin(\alpha\epsilon_1)) \cdot e^{-i\alpha\epsilon_2} \Big] \cdot dE,$$

where

$$\epsilon_1 = (E^2 - \epsilon_0^2)^{1/2}, \quad \epsilon_2 = ((E + h\omega)^2 - \epsilon_0^2)^{1/2}, \quad g(E) = \frac{E^2 + \epsilon_0^2 + h\omega E}{\epsilon_1 \epsilon_2} \;,$$

$$\alpha = \frac{R}{h v_F} \;, \qquad \Lambda(0) = \frac{m}{n e^2} \;, \qquad f = \frac{1}{1 + e^{E/k_b T}} \;, \tag{3}$$

where ϵ_0 is the energy gap at a given temperature T of superconductor, m is the effective mass, n is the electron number density, v_F is the Fermi velocity of free electrons of the metal, k_b is the Boltzmann constant. The integration over u can be easily performed:

$$\int_{-1}^{1} e^{iqRu}\cdot(1-u^2)\cdot du = 4\cdot F(qR), \quad F(x) = \frac{1}{x^2}\cdot\left(\frac{\sin x}{x} - \cos x\right)\ . \quad (4)$$

Denoting

$$C = \frac{12\pi}{c^2 h v_F \Lambda(0)}\,,$$

we can rewrite (2) as

$$K(q) = i\cdot\frac{C}{q}\int_0^\infty F(x)\cdot e^{-\frac{x}{ql}}$$

$$\times\Bigg\{\int_{\epsilon_0-h\omega}^{\epsilon_0}(1-2f(E+h\omega))\cdot(g(E)\cos(\alpha\epsilon_2)-i\cdot\sin(\alpha\epsilon_2))\cdot e^{i\alpha\epsilon_1}dE$$

$$+\int_{\epsilon_0}^{\infty}\Big[(1-2f(E+h\omega))\cdot(g(E)\cdot\cos(\alpha\epsilon_2)-i\cdot\sin(\alpha\epsilon_2))\cdot e^{i\alpha\epsilon_1}$$

$$-(1-2f(E))\cdot(g(E)\cos(\alpha\epsilon_1)+i\cdot\sin(\alpha\epsilon_1)e^{-i\alpha\epsilon_2})\Big]\ dE\Bigg\}dx \qquad (5)$$

We denote $\beta = 1/(hv_F q)$, $g_m(E) = |g(E)|$.

1. <u>The case $h\omega < 2\epsilon_0$.</u>

In the region $\epsilon_0 - h\omega < E < \epsilon_0$ we have $\epsilon_1^2 < 0,\ \epsilon_2^2 > 0$,
so $\epsilon_1 = -i|\epsilon_1|$.
In the region $\epsilon_0 < E$ we have $\epsilon_1^2 > 0$, $\epsilon_2^2 > 0$.

$$K(q) = \frac{C}{q}\Bigg\{\int_{\epsilon_0-h\omega}^{\epsilon_0}(1-2f(E+h\omega))\int_0^\infty(g_m(E)\cos(\epsilon_2\beta x)+\sin(\epsilon_2\beta x))$$

$$\times e^{-\epsilon\beta x - x/ql}F(x)dxdE + i\int_0^\infty\Big[(1-2f(E+h\omega))$$

$$\times \int_0^\infty (g(E)\cos(\epsilon_2\beta x) - i\sin(\epsilon_2\beta x)) \cdot e^{i\epsilon_1\beta x - x/(ql)} F(x)dx$$

$$-(1-2f(E)) \cdot \int_0^\infty (g(E)\cos(\epsilon_1\beta x) + i\cdot\sin(\epsilon_1\beta x))$$

$$\times e^{-i\epsilon_2\beta x - x/(ql)} \cdot F(x)\ dx \Big]\ dE \Big\}. \qquad (6)$$

2. <u>The case $h\omega > 2\epsilon_0$.</u>

In the region $\epsilon_0 - h\omega < E < -\epsilon_0$ we have ${\epsilon_1}^2 > 0,\ {\epsilon_2}^2 > 0$.
In the region $-\epsilon_0 < E < \epsilon_0$ we have $\epsilon_1^2 < 0,\ \epsilon_2^2 > 0,$
so $\epsilon_1 = -i|\epsilon_1|$.
In the region $\epsilon_0 < E$ we have $\epsilon_1^2 > 0,\ \epsilon_2^2 > 0$.
We get

$$K(q) = \frac{C}{q}\Big\{ \int_{-\epsilon_0}^{\epsilon_0} (1-2f(E+h\omega)) \cdot \int_0^\infty (g_m(E)\cos(\epsilon_2\beta x) + \sin(\epsilon_2\beta x))$$

$$\times e^{-|\epsilon_1|\beta x - x/ql} F(x)dxdE + i\Big[\int_{\epsilon_0 - h\omega}^{-\epsilon_0} (1-2f(E+h\omega))$$

$$\times \int_0^\infty (g(E)\cos(\epsilon_2\beta x) - i\sin(\epsilon_2\beta x)) \cdot e^{i\epsilon_1\beta x - x/(ql)} F(x)dxdE$$

$$+ \int_{\epsilon_0}^\infty \Big((1-2f(E+h\omega)) \cdot \int_0^\infty (g(E)\cos(\epsilon_2\beta x) - i\cdot\sin(\epsilon_2\beta x))$$

$$\times e^{i\epsilon_1\beta x - x/(ql)} \cdot F(x)\ dx - (1-2f(E)) \cdot \int_0^\infty (g(E)\cos(\epsilon_1\beta x) + i\cdot\sin(\epsilon_1\beta x))$$

$$\times e^{i\epsilon_2\beta x - x/(ql)} \cdot F(x)\ dx \Big) dE \Big] \Big\}. \qquad (7)$$

In [2] only the case $l \to \infty$ and so $R/l \to 0$ was considered. We consider the general case. Performing the integration over x in (6) and (7) it is convenient to introduce the following notations:

$$I_1 = \int_0^\infty \cos(\epsilon_2\beta x) \cdot e^{-|\epsilon_1|\beta x - x/ql} F(x)dx = -\frac{A}{2} + \frac{A^2 - \epsilon_2^2 + 1}{4} T_a + \frac{A\epsilon_2\beta}{4} L_a,$$

$$I_2 = \int_0^\infty \sin(\epsilon_2\beta x) \cdot e^{-|\epsilon_1|\beta x - x/ql} F(x)dx = \frac{\epsilon_2\beta}{2} - \frac{A\epsilon_2\beta}{2} T_a + \frac{A^2 - \epsilon_2^2 + 1}{8} L_a,$$

where

$$A = |\epsilon_1|\beta + \frac{1}{ql}; \quad T_a = arctan\left(\frac{2A}{A^2 + \beta^2\epsilon_2^2 - 1}\right);$$

$$L_a = \ln\left(\frac{A^2 + (1 + \epsilon_2\beta)^2}{A^2 + (1 - \epsilon_2\beta)^2}\right);$$

and

$$I_3 = I_{3r} + iI_{3i} = \int_0^\infty \cos(\epsilon_2\beta x) e^{i\epsilon_1\beta x - x/ql} F(x)dx,$$

$$I_{3r} = -\frac{1}{2ql} + I_{3r0}(\epsilon_1, \epsilon_2) + I_{3r0}(-\epsilon_1, \epsilon_2),$$

$$I_{3i} = \frac{\epsilon_1\beta}{2} + I_{3i0}(\epsilon_1, \epsilon_2) - I_{3i0}(-\epsilon_1, \epsilon_2),$$

$$I_4 = I_{4r} + iI_{4i} = \int_0^\infty \sin(\epsilon_2\beta x) e^{i\epsilon_1\beta x - x/ql} F(x)dx,$$

$$I_{4r} = \frac{\epsilon_2\beta}{2} + I_{4r0}(\epsilon_1, \epsilon_2) + I_{4r0}(-\epsilon_1, \epsilon_2),$$

$$I_{4i} = I_{4i0}(\epsilon_1, \epsilon_2) + I_{4i0}(-\epsilon_1, \epsilon_2),$$

where

$$I_{3r0}(\epsilon_1, \epsilon_2) = \frac{1}{8} Q(\epsilon_1, \epsilon_2) \cdot L(\epsilon_1, \epsilon_2) + \frac{1}{8} W(\epsilon_1, \epsilon_2) \cdot T(\epsilon_1, \epsilon_2),$$

$$I_{3i0}(\epsilon_1, \epsilon_2) = I_{4r0}(\epsilon_1, \epsilon_2) = \frac{1}{16} W(\epsilon_1, \epsilon_2) \cdot L(\epsilon_1, \epsilon_2) - \frac{1}{4} Q(\epsilon_1, \epsilon_2) \cdot T(\epsilon_1, \epsilon_2),$$

$$I_{4i0}(\epsilon_1, \epsilon_2) = -\frac{1}{8} Q(\epsilon_1, \epsilon_2) \cdot L(\epsilon_1, \epsilon_2) + \frac{1}{8} W(\epsilon_1, \epsilon_2) \cdot T(\epsilon_1, \epsilon_2),$$

where

$$Q(\epsilon_1, \epsilon_2) = \beta(\epsilon_2 + \epsilon_1)^2 \ ;$$

$$T(\epsilon_1, \epsilon_2) = \arctan\big(\frac{2/ql}{(1/ql)^2 + \beta^2(\epsilon_2 + \epsilon_1)^2 - 1}\big).$$

$$W(\epsilon_1, \epsilon_2) = \frac{1}{(ql)^2} + 1 - \beta^2(\epsilon_2 + \epsilon_1)^2 \ ;$$

$$L(\epsilon_1, \epsilon_2) = \ln\left(\frac{(1/ql)^2 + (1 + \beta(\epsilon_2 + \epsilon_1))^2}{(1/ql)^2 + (1 - \beta(\epsilon_2 + \epsilon_1))^2}\right).$$

In this notations the expression (6) (the case $h\omega < 2\epsilon_0$) can be rewritten as

$$K(q) = \frac{C}{q} \cdot \Bigg\{ \int_{\epsilon_0 - h\omega}^{\epsilon_0} (1 - 2f(E + h\omega)) \cdot (g_m(E) \cdot I_1 + I_2) dE +$$

$$\int_{\epsilon_0}^{\infty} \Big[(1-2f(E+h\omega)) \cdot (I_{4r} - g(E) I_{3i}) + (1-2f(E)) \cdot (I_{3i} - g(E) I_{4r}) \Big] dE.$$

$$+ i \cdot 2 \int_{\epsilon_0}^{\infty} (f(E) - f(E + h\omega))(g(E) I_{3r} + I_{4i}) dE \Bigg\} \qquad (8)$$

and the expression (7) (the case $h\omega > 2\epsilon_0$) as

$$K(q) = \frac{C}{q} \cdot \Bigg\{ \int_{\epsilon_0 - h\omega}^{-\epsilon_0} (1 - 2f(E + h\omega)) \cdot (I_{4r} - g(E) I_{3i}) dE$$

$$+ \int_{-\epsilon_0}^{\epsilon_0} (1 - 2f(E + h\omega))(g_m(E) I_1 + I_2) dE$$

$$+ \int_{\epsilon_0}^{\infty} \Big((1-2f(E+h\omega))(I_{4r} - g(E) I_{3i}) + (1-2f(E))(I_{3i} - g(E) I_{4r}) \Big) dE$$

$$+ i \Big[\int_{\epsilon_0 - h\omega}^{-\epsilon_0} (1 - 2f(E + h\omega))(g(E) I_{3r} + I_{4i}) dE$$

$$+ 2 \int_{\epsilon_0}^{\infty} (f(E) - f(E + h\omega))(g(E) I_{3r} + I_{4i}) dE \Big] \Bigg\}. \qquad (9)$$

These are our final general expressions, which can be used in numerical computations.

It is interesting to trace how the expression for the surface impedance of a metal in superconducting state $Z_s(\omega)$ transforms into that in normal state $Z_n(\omega)$ when the temperature T is increased up to the critical temperature of the superconducting transition T_c, that is when the superconducting energy gap $\epsilon_0(T \to 0) \to 0$. Putting $\epsilon_0 = 0$ in (5) we get

$$K(q) = \frac{C \cdot h\omega}{q} \cdot i \cdot \int_0^\infty F(x) \cdot e^{-\frac{x}{ql} \cdot (1+i\omega\tau)} dx$$

$$= i \cdot \frac{\alpha_{RS}}{l^2(1+i\omega\tau)} \cdot \frac{1}{t^3} \cdot \Big(-t + (t^2+1) \arctan t \Big).$$

Here $\tau = l/v_F$ is the relaxation time,

$$\alpha_{RS} = \frac{3}{2} \cdot \left(\frac{l}{\delta} \right)^2 , \quad t = \frac{ql}{1+i\omega\tau} \quad \delta = \frac{c}{\sqrt{2\pi\sigma\omega}},$$

δ is the skin depth .

Putting this $K(q)$ into the (1s) we get

$$Z_s(\omega) = i \cdot \frac{8\omega l}{c^2} \cdot \frac{1}{1+i\omega\tau} \cdot \int_0^\infty \frac{dt}{t^2 + \xi\kappa(t)} , \tag{10}$$

where

$$\xi = i \cdot \frac{\alpha_{RS}}{(1+i\omega\tau)^3} , \qquad \kappa(t) = \frac{1}{t^3} \cdot \Big(-t + (1+t^2) \arctan t \Big) .$$

This expressions coincide with the expressions (21,40,41) of [4]. Taking into account the relation

$$\arctan t = \frac{1}{2i} \ln \left(\frac{1+it}{1-it} \right) ,$$

we can rewrite (10) as

$$Z_s(\omega) = i \cdot \frac{8\omega l}{c^2} \cdot \int\limits_0^{\infty} \frac{dt}{t^2 + \frac{i\alpha_{RS}}{1+i\omega\tau} \cdot K_{RS}(\frac{it}{1+i\omega\tau})}$$

where

$$K_{RS}(s) = \frac{1}{s^3} \cdot \left(2s - (1-s^2) \cdot \ln\left(\frac{1+s}{1-s}\right)\right)$$

This coincide with the expressions (21, 29, 30) of [4].

In the extreme anomalous limit, when the field penetration depth is small compared with the coherence length and with free path length one may set $\alpha = R/hv_0 = 0, \qquad ql >> 1$ [2, 3]. Using the relations ($F(x)$ is defined by(4))

$$\int\limits_0^{\infty} F(x)dx = \frac{\pi}{4}, \qquad \sigma = \frac{ne^2 l}{mv_F},$$

and setting $\alpha = 0$ in (6) and (7) we get for the case

$$h\omega < 2\epsilon_0$$

$$K(q) = \frac{3\pi^2\sigma\omega}{c^2 ql} \cdot \Big\{ \frac{1}{h\omega} \int\limits_{\epsilon_0 - h\omega}^{\epsilon_0} (1 - 2f(E+h\omega))g_m(E)dE$$

$$+i \cdot \frac{2}{h\omega} \int\limits_{\epsilon_0}^{\infty} (f(E) - f(E+h\omega)) \cdot g(E)dE \Big\} = \frac{3\pi^2\sigma\omega}{c^2 ql} \cdot \left(\frac{\sigma_2}{\sigma} + i\frac{\sigma_1}{\sigma}\right). \quad (12)$$

and for the case

$$h\omega > 2\epsilon_0$$

$$K(q) = \frac{3\pi^2\sigma\omega}{c^2 ql} \cdot \Big\{ \frac{1}{h\omega} \int\limits_{-\epsilon_0}^{\epsilon_0} (1 - 2f(E+h\omega))g_m(E)dE +$$

$$i\cdot\frac{1}{h\omega}\cdot\Big[2\int\limits_{\epsilon_0}^{\infty}(f(E)-f(E+h\omega))\cdot g(E)dE+\int\limits_{\epsilon_0-h\omega}^{-\epsilon_0}(1-2f(E+h\omega))g(E)dE\Big]\Big\}$$

$$=\frac{3\pi^2\sigma\omega}{c^2ql}\cdot\left(\frac{\sigma_2}{\sigma}+i\frac{\sigma_1}{\sigma}\right). \qquad (13)$$

The relations (12) and (13) are definitions of the functions σ_2/σ and σ_1/σ for both the cases.

Inserting the expressions (12) or (13) into (1s) and performing a simple integration we get for the case of specular reflection:

$$Z_{ss}=\frac{8}{9}\cdot\left(\sqrt{3}\frac{\pi\omega^2 l}{c^4\sigma}\right)^{1/3}\cdot\frac{(1+i\sqrt{3})}{(\sigma_1/\sigma-i\sigma_2/\sigma)^{1/3}}=\frac{Z_{Ns}}{(\sigma_1/\sigma-i\sigma_2/\sigma)^{1/3}}\ .$$

And inserting the expressions (8) or(9) into (1d) we get for the case of diffusive reflection:

$$Z_{sd}=\left(\sqrt{3}\frac{\pi\omega^2 l}{c^4\sigma}\right)^{1/3}\cdot\frac{(1+i\sqrt{3})}{(\sigma_1/\sigma-i\sigma_2/\sigma)^{1/3}}=\frac{Z_{Nd}}{(\sigma_1/\sigma-i\sigma_2/\sigma)^{1/3}}\ .$$

Comparing Z_{Ns} and Z_{Nd}, respectively, with the expressions (44) and (45) of [4], we see that these are just the anomalous surface impedances of normal metal with free path length l at T_c respectively for specular and diffusive reflections of electrons at the surface.

We can get the same expressions for Z_{Ns} and Z_{Nd} directly from (13) in the limit $\epsilon_0(T\to T_c)\to 0$ (then we have the metal in normal state) and

$$K(q)=i\cdot\frac{3\pi^2\sigma\omega}{c^2ql}. \qquad (14)$$

Then it follows from (13) that $\sigma_1/\sigma = 1$, $\sigma_2/\sigma=0$, as it should be. Inserting (14) into (1s) and (1d) we get, respectively, Z_{Ns}, and Z_{Nd}. In the limit of high frequencies $(h\omega >> \epsilon_0)$ the surface impedance of the metal in superconducting state $Z_s(\omega)$ approach the surface impedance of the metal in normal state $Z_N(\omega)$ at T_c . But $Z_N(\omega)$ for the frequencies $\omega\tau > 1$ essentially deviates from the anomalous surface impedances Z_{Ns}, and Z_{Nd}. . So all the results obtained for the so called extreme anomalous limit are applicable only for frequencies $\omega\tau < 1$.

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