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VARIATION OF RADIATION LENGTH
DUE TO LPM EFFECT

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Abstract

The photon emission intensity spectrum is calculated taking into account influence of multiple scattering (the LPM effect) under conditions of recent CERN SPS experiment. It is shown that the theory quite satisfactory describes data. The integral characteristics: the radiation length and total probability of photon emission are analyzed under influence of the LPM effect, and the asymptotic expansions of these characteristics are derived.

1 Introduction

When a charged particle is moving in a medium it scatters on atoms. With probability $\sim \alpha$ this scattering is accompanied by a radiation. At high energy the radiation process occurs over a rather long distance, known as the *formation length* l_c (see e.g.[1]):

$$l_c = \frac{l_0}{1 + \gamma^2 \vartheta_c^2}, \quad l_0 = \frac{2\varepsilon\varepsilon'}{m^2\omega}, \quad (1.1)$$

where ω is the energy of emitted photon, $\varepsilon(m)$ is the energy (the mass) of a particle, $\varepsilon' = \varepsilon - \omega$, ϑ_c is the characteristic angle of photon emission, the system $\hbar = c = 1$ is used.

Landau and Pomeranchuk were the first who showed that if the formation length of bremsstrahlung becomes comparable to the distance over which the multiple scattering becomes important, the bremsstrahlung will be suppressed [2]. Migdal [3] developed the quantitative theory of this phenomenon.

A new interest to the theory of the LPM effect is connected with a very successful series of experiments performed at SLAC [4]. In these experiments the cross section of the bremsstrahlung of soft photons with energy from 200 keV to 500 MeV from electrons with energy 8 GeV and 25 GeV is measured with an accuracy of the order of a few percent. Both LPM and dielectric suppression are observed and investigated. The logarithmic binning of photon spectrum was used with 25 bins per decade. These experiments were the challenge for the theory since in all the mentioned papers calculations are performed to logarithmic accuracy which is not enough for description of the new experiment. The contribution of the Coulomb corrections (at least for heavy elements) is larger than experimental errors and these corrections should be taken into account.

Recently new study of LPM effect at higher energies of electrons ($\varepsilon = 149, 207$ and 287 GeV), where the effect has influence upon much wider part of spectrum comparing with $\varepsilon = 25$ GeV, was performed in the H2 beam line

of the CERN SPS [5],[6]. The logarithmic binning of photon spectrum was used with 25 bins per decade as at SLAC.

We developed the new approach to the theory of the Landau-Pomeranchuk-Migdal (LPM) effect [7]. In this paper the cross section of the bremsstrahlung process in the photon energies region where the influence of the LPM is very strong was calculated with a term $\propto 1/L$, where L is characteristic logarithm of the problem, and with the Coulomb corrections taken into account. In the photon energy region, where the LPM effect is "turned off", the obtained cross section gives the exact Bethe-Maximon cross section (within power accuracy and with the Coulomb corrections). This important feature was absent in the previous calculations. The LPM effect in a thin target were the interference effects and boundary photon emission is of special interest was analyzed in [8]. The probability of multiphoton emission is enhanced at high energy [9]. Correspondingly this effect is very important for comparison of theory prediction and data. The influence of LPM effect on integral characteristics of bremsstrahlung was considered in [10]. The other approaches to the LPM effect theory see e.g. in [11], [12], the recent review is given in [13].

In Sec.2 the theory predictions are compared with the recent CERN SPS data [5],[6]. It is shown that the theory quite satisfactory describes data. In Sec.3 the variation of the radiation length due to multiple scattering is discussed. In the region where the LPM effect is weak ($\varepsilon \ll \varepsilon_e$) the asymptotic expressions for the radiation length and the photon emission probability are derived taking into account both decomposition over $\varepsilon/\varepsilon_e$ and over $1/L_1$, L_1 is the characteristic logarithm of the problem. Appendices A and B contain details of derivation for radiation length and Appendix C for the photon emission probability. In Appendix D the structure of series over $\varepsilon/\varepsilon_e$ and over $1/L_1$ is analyzed for $\varepsilon \ll \varepsilon_e$.

2 Influence of the multiple scattering on the bremsstrahlung spectrum

The spectral radiation intensity obtained in [7], Eq.(2.39) (see also Sec.3 in [10]) is valid for any energy and has the form

$$dI = \omega dW = \frac{\alpha m^2 x dx}{2\pi(1-x)} \text{Im} \left[\Phi(\nu) - \frac{1}{2L_c} F(\nu) \right], \quad x = \frac{\omega}{\varepsilon}, \quad (2.1)$$

where

$$\begin{aligned}
\Phi(\nu) &= \int_0^\infty dz e^{-it} \left[r_1 \left(\frac{1}{\sinh z} - \frac{1}{z} \right) - i\nu r_2 \left(\frac{1}{\sinh^2 z} - \frac{1}{z^2} \right) \right] \\
&= r_1 \left(\ln p - \psi \left(p + \frac{1}{2} \right) \right) + r_2 \left(\psi(p) - \ln p + \frac{1}{2p} \right), \\
F(\nu) &= \int_0^\infty \frac{dz e^{-it}}{\sinh^2 z} [r_1 f_1(z) - 2ir_2 f_2(z)], \\
t &= \frac{z}{\nu}, \quad r_1 = x^2, \quad r_2 = 1 + (1-x)^2,
\end{aligned} \tag{2.2}$$

where $z = \nu t$, $p = i/(2\nu)$, $\psi(x)$ is the logarithmic derivative of the gamma function. The functions $f_1(z)$ and $f_2(z)$ are defined by following expressions

$$\begin{aligned}
f_1(z) &= \left(\ln \varrho_c^2 + \ln \frac{\nu}{i} - \ln \sinh z - C \right) g(z) - 2 \cosh z G(z), \\
f_2(z) &= \frac{\nu}{\sinh z} \left(f_1(z) - \frac{g(z)}{2} \right), \quad g(z) = z \cosh z - \sinh z, \\
G(z) &= \int_0^z (1 - y \coth y) dy \\
&= z - \frac{z^2}{2} - \frac{\pi^2}{12} - z \ln(1 - e^{-2z}) + \frac{1}{2} \text{Li}_2(e^{-2z}),
\end{aligned} \tag{2.3}$$

here $\text{Li}_2(x)$ is the Euler dilogarithm, $c = 0.577216\dots$ is Euler's constant. Use of the last representation of function $G(z)$ simplifies the numerical calculation. The crucial parameter ν is

$$\begin{aligned}
\nu^2 &= i\nu_0^2, \quad \nu_0^2 = |\nu|^2 \simeq \nu_1^2 \left(1 + \frac{\ln \nu_1}{L_1} \vartheta(\nu_1 - 1) \right), \quad \nu_1^2 = \frac{\varepsilon}{\varepsilon_e} \frac{1-x}{x}, \\
\varepsilon_e &= m (8\pi Z^2 \alpha^2 n_a \lambda_c^3 L_1)^{-1}, \quad L_c \simeq L_1 \left(1 + \frac{\ln \nu_1}{L_1} \vartheta(\nu_1 - 1) \right), \quad L_1 = \ln \frac{a_{s2}^2}{\lambda_c^2}, \\
\frac{a_{s2}}{\lambda_c} &= 183Z^{-1/3} e^{-f}, \quad f = f(Z\alpha) = (Z\alpha)^2 \sum_{k=1}^{\infty} \frac{1}{k(k^2 + (Z\alpha)^2)},
\end{aligned} \tag{2.4}$$

where Z is the charge of the nucleus, n_a is the number density of atoms in the medium, $\lambda_c = 1/m = (\hbar/mc)$ is the electron Compton wavelength. Here ε_e is a characteristic parameter of medium, starting from this energy the multiple scattering distorts the whole spectrum of bremsstrahlung including its hard part: for iridium $\varepsilon_e=2.27$ TeV, for gold $\varepsilon_e=2.6$ TeV, for tungsten $\varepsilon_e=2.73$ TeV, for tantalum $\varepsilon_e=3.18$ TeV, for lead $\varepsilon_e=4.38$ TeV.

In the case $\varepsilon \ll \varepsilon_e$ the LPM effect manifests itself when

$$\nu_1(x_c) = 1, \quad x_c = \frac{\varepsilon}{\varepsilon_e + \varepsilon} \simeq \frac{\varepsilon}{\varepsilon_e}. \tag{2.5}$$

In this case in the hard part of spectrum ($1 \geq x \gg x_c$, $\nu_1^2 \simeq x_c/x \ll 1$) one has (see Appendices A and B)

$$\begin{aligned} \frac{dI}{dx} &= \frac{\varepsilon}{L_{rad}^0} \left\{ x^2 + \frac{4(1-x)}{3} + \frac{2(1-x)}{9L_1} - \frac{\varepsilon^2}{\varepsilon_e^2} \frac{(1-x)^2}{x^2} \right. \\ &\quad \times \left[\left(\frac{64}{63} - \frac{15272}{2205} \frac{1}{L_1} \right) (1-x) + \left(1 - \frac{5017}{1800} \frac{1}{L_1} \right) x^2 \right] \left. \right\}, \\ \frac{1}{L_{rad}^0} &= \frac{2Z^2 \alpha^3 n_a L_1}{m^2} = \frac{\alpha}{4\pi} \frac{m}{\varepsilon_e \lambda_c}, \quad \frac{1}{L_{rad}^{BM}} = \frac{1}{L_{rad}^0} \left(1 + \frac{1}{9L_1} \right), \quad (2.6) \end{aligned}$$

here L_{rad}^{BM} is the Bethe-Maximon radiation length. Note that if neglect here the terms $\propto \varepsilon^2/\varepsilon_e^2$ we obtain the Bethe-Maximon intensity spectrum.

In the recent CERN SPS experiment the iridium target with thickness $l=0.128$ mm ($l/L_{rad} = 4.36 \pm 0.10\%$, L_{rad} is the radiation length) [5] and tantalum target [6] with thickness $l/L_{rad} = 4.45 \pm 0.10\%$ were used. Photons with energy $2 \text{ GeV} < \omega < \varepsilon$ were detected in a lead glass calorimeter.

Obtained experimental data should be recalculated:

$$\left(\frac{d\varepsilon}{d\omega} \right)_{exp} = \frac{l}{L_{rad}} \frac{1}{k} \left(\frac{dN}{d \ln \omega} \right)_{exp}. \quad (2.7)$$

Because photon energies were histogrammed logarithmically, using 25 bins per decade of energy, one has for the coefficient k

$$\begin{aligned} k_h &= \frac{\omega_{max} - \omega_{min}}{\omega_{min}} = e^s - 1 = 0.096, \\ k_m &= 2 \frac{\omega_{max} - \omega_{min}}{\omega_{max} + \omega_{min}} = 2 \frac{e^s - 1}{e^s + 1} = 0.092, \\ k_l &= \frac{\omega_{max} - \omega_{min}}{\omega_{max}} = 1 - e^{-s} = 0.88, \\ s &= \ln \frac{\omega_{max}}{\omega_{min}} = \frac{\ln 10}{25} = 0.092, \quad (2.8) \end{aligned}$$

depending on the normalization point within bin. Here we will use k_m .

Since only high energy photons ($\omega \geq 2 \text{ GeV}$) were measured, no boundary effects were observed. So, only the pure LPM effect was studied. For iridium $\varepsilon_e=2.27 \text{ TeV}$ and from one has Eq.(2.5) that the characteristic photon energy $\omega_c(\varepsilon)$ for which the LPM effect is well manifests itself is $\omega_c(287 \text{ GeV})=32 \text{ GeV}$ and $\omega_c(207 \text{ GeV})=19 \text{ GeV}$, while for tantalum

$\varepsilon_e=3.18$ TeV and $\omega_c(287 \text{ GeV})=26$ GeV. The results of calculations for iridium (the initial electron energy $\varepsilon=287$ GeV and $\varepsilon=207$ GeV) and for tantalum (the initial electron energy $\varepsilon=287$ GeV) are shown in Figs.1-3. The curve 1 is the Bethe-Maximon intensity spectrum (see e.g.Eq.(2.6)). The curves 2,3 are calculated using Eqs. (2.1) and (2.2). The curve 2 presents the main term (with the function $\Phi(\nu)$), the curve 3 presents the correction term (with the function $F(\nu)$). It should be noted that the prediction of our theory (sum of previous terms, curve 4) in the hard end of spectrum coincide with the

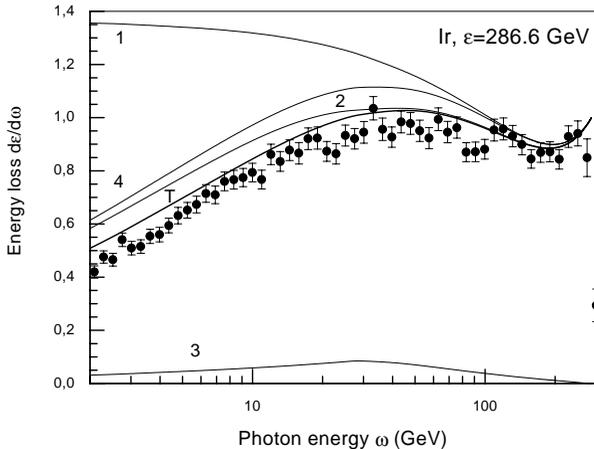


Figure 1: The energy loss spectrum $\frac{d\varepsilon}{d\omega}$ in units $\frac{1}{L_{rad}^0}$ in iridium target with thickness $l=0.128$ mm ($l/L_{rad}^{BM} = 4.36\%$) for the initial electrons energy is $\varepsilon = 286.6$ GeV. The Coulomb corrections are included: *curve 1* is the Bethe-Maximon spectrum, *curve 2* is the contribution of the main term describing LPM effect; *curve 3* is the correction term; *curve 4* is the sum of two previous contributions; *curve T* is the final theory prediction with regard for the reduction factor (the multiphoton effects). Experimental data from [5], [6].

Bethe-Maximon curve within the accuracy better than 10^{-3} . It is seen that given values $\omega_c(\varepsilon)$ show the scale where LPM effect becomes essential. For used thickness of target the multi-photon effects are very essential. The reduction factor f in this case it is convenient to calculate using the following general expression (Eq.(3.4) of [9]):

$$\frac{d\varepsilon}{d\omega} = \omega \frac{d\omega}{d\omega} f, \quad f = \exp \left[- \int_{\omega}^{\infty} \frac{d\omega}{d\omega_1} d\omega_1 \right] \left(1 + O \left(\omega \frac{d\omega}{d\omega} \right) \right), \quad (2.9)$$

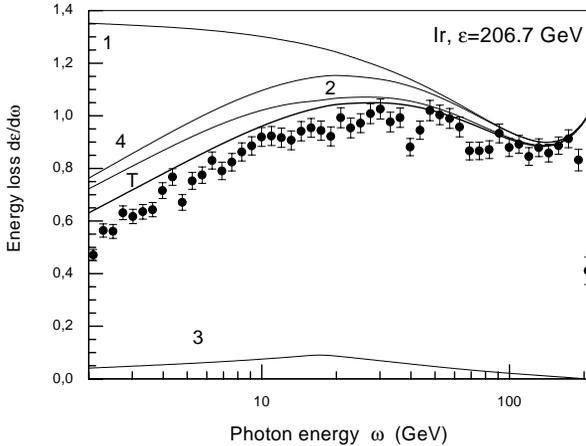


Figure 2: The same as in Fig.1 but for the initial electrons energy $\varepsilon = 206.7$ GeV.

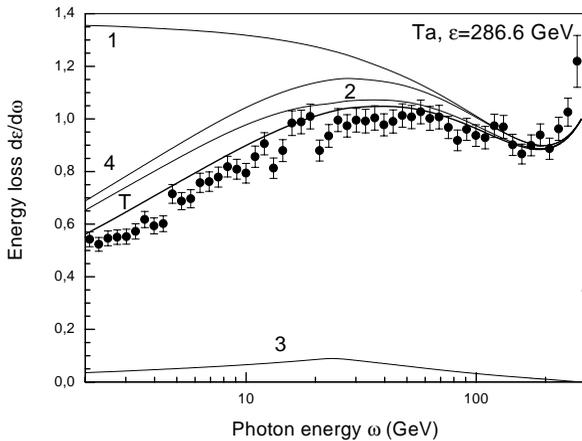


Figure 3: The same as in Fig.1 but for tantalum target with thickness ($l/L_{rad}^{BM} = 4.45\%$).

where the main term (with $\Phi(\nu)$) of spectrum Eq.(2.1) is substituted. The results obtained are in a good agreement with Eqs.(2.11) and (2.26) of [9]. The reduction factor f in iridium for three used energies is presented in Fig.4. It follows from Fig.4 that if the electron energy decreases the reduction factor

f diminishes as well. The final prediction, with the reduction factor taken into account for used target thicknesses, present the curves T . The data are recalculated according with Eqs.(2.7),(2.8) using the coefficient k_m . In Figs.1-3 one can see that for energy 287 GeV there is quite satisfactory agreement of theory with data for both iridium and tantalum, for energy 207 GeV in iridium the agreement is somewhat less satisfactory.

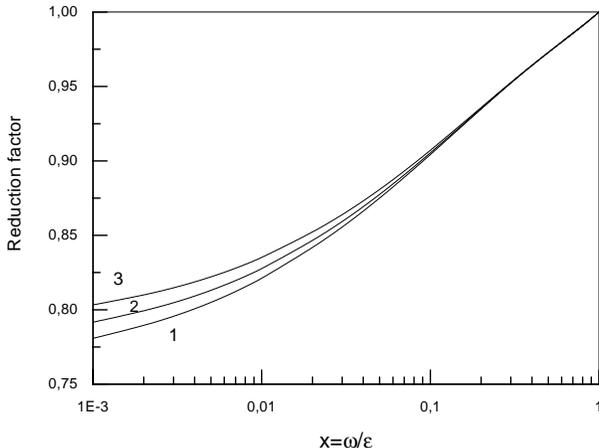


Figure 4: The reduction factor for iridium target with thickness $l = 0.128 \text{ mm}$ (4.36% L_{rad}) versus $x = \omega/\varepsilon$. The curves 1,2,3 are for energies 149 GeV, 207 GeV and 287 GeV correspondingly.

3 The LPM radiation length and the average number of emitted photons at $\varepsilon \ll \varepsilon_e$

The local radiation length L_{rad} is defined by equation

$$\frac{d\varepsilon}{dt} = -I(\varepsilon) = -\frac{\varepsilon}{L_{rad}}. \quad (3.1)$$

In the absence of LPM effect the value L_{rad} doesn't dependent on energy and defined by Eq.(2.6). In this case, after averaging Eq.(3.1) with the electron distribution function over energy $f(\varepsilon, t)$ one obtains the closed equation for the mean energy loss which has the solution $\langle \varepsilon(t) \rangle = \varepsilon_0 \exp(-t/L_{rad})$.

For the first photon emission according with Eq.(3.1) $L_{rad} = \varepsilon/I(\varepsilon)$. The dependence on the energy of the function $(I(\varepsilon)/\varepsilon)L_{rad}^{BM} = L_{rad}^{BM}/L_{rad}$ for

iridium (curve 1) and lead (curve 2) is given in Fig.5. The $I(\varepsilon)$ is the integrated over x Eq.(2.1). The relative value of correction (the term with $F(\nu)$) to the main term (with $\Phi(\nu)$) depends on energy. It attains the maximum $\sim 5\%$ near $\varepsilon \sim \varepsilon_e$ for both media. So, the value of L_{rad} for the energy $\varepsilon = 3$ TeV increases 1.33 times in Ir and in 1.16 times in Pb, while for the energy $\varepsilon = 10$ TeV the corresponding figures are 1.58 and 1.38.

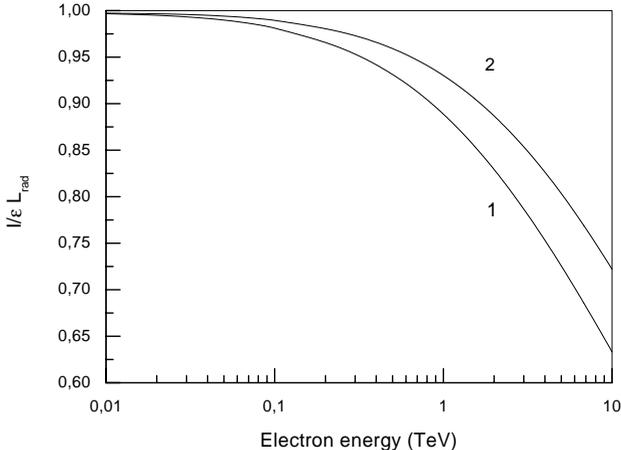


Figure 5: The relative energy losses of electron per unit time including the contribution of the correction term $(I(\varepsilon)/\varepsilon)L_{rad}^{BM} = L_{rad}^{BM}/L_{rad}$ in iridium (curve 1) and in lead (curve 2) vs the initial energy of electron.

In the region where the LPM effect is weak (see Appendices A and B)

$$\frac{1}{L_{rad}} = \frac{1}{L_{rad}^0} \left\{ 1 + \frac{1}{9L_1} - \frac{4\pi}{15} \frac{\varepsilon}{\varepsilon_e} + \frac{64}{21} \frac{\varepsilon^2}{\varepsilon_e^2} \left(\ln \frac{\varepsilon_e}{\varepsilon} - 2.040 \right) + \frac{1}{L_1} \left[\frac{182\pi}{225} \frac{\varepsilon}{\varepsilon_e} - \frac{15272}{735} \frac{\varepsilon^2}{\varepsilon_e^2} \left(\ln \frac{\varepsilon_e}{\varepsilon} - 2.577 \right) \right] \right\}. \quad (3.2)$$

The first term of expansion over $\varepsilon/\varepsilon_e$ (not over $1/L_1$) was found in [10].

The accuracy of Eq.(3.2) is defined by a several essentially different factors. The first is the relative value of discarded terms $O(\varepsilon^3/\varepsilon_e^3)$ in Eqs.(A.13), B.15). The second is the substitution of L_c by L_1 ($\varrho_c = 1$) in Eqs.(2.1) - (2.4) for the whole spectrum, the relative accuracy of this substitution is $O(\varepsilon/(\varepsilon_e L_1^2))$. The third is due to fact that in the initial formula Eq.(2.1) the

terms $\propto 1/L_c^2$ were rejected. However, as is shown in in Appendix D, in the region $\varepsilon \ll \varepsilon_e$ the corrections $\propto 1/L_1^2$ are beginning with the terms contained $\varepsilon/\varepsilon_e$. So, in this region, the relative accuracy in the formula Eq.(2.1) is also $O(\varepsilon/(\varepsilon_e L_1^2))$.

One can see from Eq.(3.2), that in a decomposition over $\varepsilon/\varepsilon_e$ one have to take into account the terms $\propto 1/L_1$ because of relatively large numerical values of the coefficients of decomposition. Moreover, for heavy elements $L_1 \simeq (6 \div 7)$ taking into account of term $\propto 1/L_1$ results in substantial change of numerical value of coefficients at the given degree of $\varepsilon/\varepsilon_e$. Particularly this is important for term containing $\ln(\varepsilon_e/\varepsilon)$. As a result, there is essential compensation inside the coefficients of decomposition. This permit to use the decomposition Eq.(3.2) at relatively large energies. At $\varepsilon/\varepsilon_e \leq 1/10$ an error in the correction terms doesn't exceed 10%.

As illustration we present the radiation length in iridium ($L_1 = 6.9225$, $\varepsilon_e = 2.27$ TeV, $L_{rad}^{BM}(\text{Ir})=2.91$ mm):

$$\frac{1}{L_{rad}(\text{Ir})} = \frac{1}{L_{rad}^{BM}(\text{Ir})} \left(1 - 0.464 \frac{\varepsilon}{\varepsilon_e} + 0.045 \frac{\varepsilon^2}{\varepsilon_e^2} \ln \frac{\varepsilon_e}{\varepsilon} + 1.492 \frac{\varepsilon^2}{\varepsilon_e^2} \right). \quad (3.3)$$

It should be noted that after substitution of Eq.(3.3) into Eq.(3.1) and averaging with the distribution function $f(\varepsilon, t)$ the higher moments of ε appear. Because of this, equation (3.1) ceases to be closed. We will discuss this item elsewhere.

The mean value of photons emitted by electron is also of evident interest. Particularly, at $\omega \ll \omega_c \simeq \varepsilon^2/\varepsilon_e$ this value defines mainly the exponent in Eq.(2.9). At energy where the LPM effect is rather weak ($\varepsilon \ll \varepsilon_e$) the radiation probability per unit time integrated over whole photon spectrum is (see Appendix C)

$$W(\varepsilon) = \frac{dw}{dt} = \frac{\alpha m}{3\pi\varepsilon_e\lambda_c} \left[\ln \frac{\varepsilon_e}{\varepsilon} + 1.959 + \frac{2\pi}{5} \frac{\varepsilon}{\varepsilon_e} + \frac{1}{L_1} \left(\frac{1}{6} \ln \frac{\varepsilon_e}{\varepsilon} + 3.574 - \frac{91\pi}{75} \frac{\varepsilon}{\varepsilon_e} \right) \right]. \quad (3.4)$$

It is seen that allowing for terms $\propto 1/L_1$ changes essentially non-logarithmic terms in Eq.(3.4).

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A Appendix

We consider here the asymptotic behavior of the total intensity of radiation I_0 in the region $\varepsilon \ll \varepsilon_e$ where the LPM effect is weak and one can put in Eqs.(2.1)-(2.4) $\nu^2 = i\nu_1^2$, $\varrho_c = 1$, $L_c = L_1$. This substitution will manifest itself only starting from terms of the order $\frac{1}{L_1^2} \frac{\varepsilon}{\varepsilon_e} I_0$. Then in expressions for $\Phi(\nu)$ and $F(\nu)$ in Eq.(2.2) we integrate by parts the terms $\propto r_1$ and make the substitution of variable $t \rightarrow -it$. As a result we obtain

$$I = \frac{\varepsilon}{L_{rad}^0} \left(J_m + \frac{1}{L_1} J_v \right), \quad (\text{A.1})$$

where

$$J_m = 2\text{Re} \int_0^1 dx \int_0^\infty dt e^{-t} [x^2 \varphi_1(z) - 2(1-x)\varphi_2(z)];$$

$$\varphi_1(z) = \frac{1}{\cosh z + 1}, \quad \varphi_2(z) = \frac{1}{\sinh^2 z} - \frac{1}{z^2}, \quad (\text{A.2})$$

and

$$J_v = -\text{Re} \int_0^1 dx \int_0^\infty dt e^{-t} [x^2 l_1(z, t) + 4(1-x)l_2(z, t)];$$

$$l_{1,2} = a_{1,2}(z)(\ln t + C) + b_{1,2}(z),$$

$$a_1(z) = \varphi_1(z) - \frac{z}{\sinh z}, \quad a_2(z) = \frac{z \coth z - 1}{\sinh^2 z},$$

$$b_1(z) = a_1(z) \ln \frac{\sinh z}{z} + \frac{1}{\cosh z + 1} \left(1 - z \coth z + \frac{2}{\sinh z} G(z) \right),$$

$$b_2(z) = a_2(z) \ln \frac{\sinh z}{z} + \frac{1}{2 \sinh^2 z} (4G(z) \coth z + z \coth z - 1), \quad (\text{A.3})$$

here the function $G(z)$ is defined in Eq.(2.3) and

$$\beta^2 = i \frac{\varepsilon_e}{\varepsilon}, \quad z = \frac{t}{\beta y}, \quad y = \sqrt{\frac{x}{1-x}}. \quad (\text{A.4})$$

In Eq.(A.2) one can expand the function $\varphi_1(z)$ up to terms $\propto z^4$. We find

$$J_{m1} \simeq 2 \int_0^1 dx x^2 \int_0^\infty dt e^{-t} \left(\varphi_1(0) + \frac{\varphi_1^{(4)}(0) t^4}{\beta^4 y^4 4!} \right) = \frac{2}{3} \left[\varphi_1(0) - \frac{\varepsilon^2}{\varepsilon_e^2} \varphi_1^{(4)}(0) \right]. \quad (\text{A.5})$$

In the integral with function $\varphi_2(z)$ in Eq.(A.2) we perform twice integration by parts over t . We find

$$\begin{aligned}
J_{m2} &= -4\text{Re} \int_0^1 dx(1-x) \left[\varphi_2(0) + \frac{1}{\beta^2 y^2} \int_0^\infty dt e^{-t} \varphi_2^{(2)}(z) \right] \\
&= -2\varphi_2(0) - 4 \frac{\varepsilon}{\varepsilon_e} S, \\
S &= \text{Im} \int_0^1 \frac{(1-x)^2 dx}{x} \int_0^\infty dt e^{-t} \varphi_2^{(2)}(z). \tag{A.6}
\end{aligned}$$

To calculate S we choose the value y_0 in such way that $y_0 \ll 1$ and $|\beta|y_0 \gg 1$ and divide the integral over x into two. The first one is

$$\begin{aligned}
S_1 &= \text{Im} \int_{x_0}^1 \frac{(1-x)^2 dx}{x} \int_0^\infty dt e^{-t} \varphi_2^{(2)}(z) \\
&\simeq \text{Im} \int_{x_0}^1 \frac{(1-x)^2 dx}{x} \frac{1}{\beta^2 y^2} \varphi_2^{(4)}(0) \\
&\simeq -\frac{1}{|\beta|^2} \left(\frac{1}{x_0} - 1 + 3 \ln x_0 + \frac{5}{2} \right) \varphi_2^{(4)}(0) \\
&\simeq -\frac{1}{|\beta|^2} \left(\frac{1}{y_0^2} + 6 \ln y_0 + \frac{5}{2} \right) \varphi_2^{(4)}(0). \tag{A.7}
\end{aligned}$$

In the second integral we pass to the variable y :

$$\begin{aligned}
S_2 &= 2\text{Im} \int_0^{y_0} \frac{dy}{y(1+y^2)^3} \int_0^\infty dt e^{-t} \varphi_2^{(2)}(z) \\
&\simeq 2\text{Im} \int_0^{y_0} (1-3y^2) \frac{dy}{y} \int_0^\infty dt e^{-t} \varphi_2^{(2)}(z). \tag{A.8}
\end{aligned}$$

Let us consider the integral

$$\begin{aligned}
S_{21} &= 2\text{Im} \int_0^{y_0} \frac{dy}{y} \int_0^\infty dt e^{-t} \varphi_2^{(2)} \left(\frac{t}{\beta y} \right) = \frac{1}{i} \int_{\beta^* y_0}^{\beta y_0} \frac{dy}{y} \int_0^\infty dt e^{-t} \varphi_2^{(2)} \left(\frac{t}{y} \right) \\
&\simeq \frac{1}{i} \int_{\beta^* y_0}^{\beta y_0} \frac{dy}{y} \left(\varphi_2^{(2)}(0) + \frac{1}{y^2} \varphi_2^{(4)}(0) \right) \\
&= \frac{1}{i} \left[\ln \frac{\beta}{\beta^*} \varphi_2^{(2)}(0) - \frac{1}{2} \left(\frac{1}{\beta^2 y_0^2} - \frac{1}{\beta^{*2} y_0^2} \right) \varphi_2^{(4)}(0) \right] \\
&= \frac{\pi}{2} \varphi_2^{(2)}(0) + \frac{1}{|\beta|^2 y_0^2} \varphi_2^{(4)}(0). \tag{A.9}
\end{aligned}$$

The remaining integral in Eq.(A.8) is

$$\begin{aligned}
S_{22} &= -6\text{Im} \int_0^{y_0} y dy \int_0^\infty dt e^{-t} \varphi_2^{(2)}(z) \\
&= \frac{6}{|\beta|^2} \text{Re} \int_0^{\beta y_0} \frac{dy}{y} \int_0^\infty dt e^{-t} \varphi_2^{(4)}\left(\frac{t}{y}\right) \\
&= \frac{6}{|\beta|^2} \text{Re} \left[\ln(\beta y_0) \int_0^\infty dt e^{-t} \varphi_2^{(4)}\left(\frac{t}{\beta y_0}\right) \right. \\
&\quad \left. + \int_0^{\beta y_0} \ln y dy \int_0^\infty \frac{t dt}{y^2} e^{-t} \varphi_2^{(5)}\left(\frac{t}{y}\right) \right] \\
&\simeq \frac{6}{|\beta|^2} \text{Re} \left[\ln(\beta y_0) \varphi_2^{(4)}(0) + \int_0^\infty \ln y dy \int_0^\infty z e^{-zy} \varphi_2^{(5)}(z) dz \right] \\
&= \frac{6}{|\beta|^2} \text{Re} \left[\ln(\beta y_0) \varphi_2^{(4)}(0) + \int_0^\infty dy \int_0^\infty (\ln y - \ln z) e^{-y} \varphi_2^{(5)}(z) dz \right] \\
&= \frac{6}{|\beta|^2} \text{Re} \left[(\ln(\beta y_0) + C) \varphi_2^{(4)}(0) - \int_0^\infty \ln z \varphi_2^{(5)}(z) dz \right]. \quad (\text{A.10})
\end{aligned}$$

Summing S_1 , S_{21} and S_{22} we obtain for S

$$S = \frac{\pi}{2} \varphi_2^{(2)}(0) + 3 \frac{\varepsilon}{\varepsilon_e} \left[\left(\ln \frac{\varepsilon_e}{\varepsilon} + 2C - \frac{5}{6} \right) \varphi_2^{(4)}(0) - 2 \int_0^\infty \ln z \varphi_2^{(5)}(z) dz \right]. \quad (\text{A.11})$$

Substituting Eq.(A.11) into J_{m2} , adding with J_{m1} Eq.(A.5) and taking into account that

$$\begin{aligned}
\varphi_1(0) = \varphi_1^{(4)}(0) &= \frac{1}{2}, \quad \varphi_2(0) = -\frac{1}{3}, \quad \varphi_2^{(2)}(0) = \frac{2}{15}, \quad \varphi_2^{(4)}(0) = -\frac{16}{63}, \\
\int_0^\infty \ln z \varphi_2^{(5)}(z) dz &= -0.286, \quad (\text{A.12})
\end{aligned}$$

we obtain

$$\begin{aligned}
J_m &= 1 - \frac{4\pi}{15} \frac{\varepsilon}{\varepsilon_e} + \frac{64}{21} \frac{\varepsilon^2}{\varepsilon_e^2} \left(\ln \frac{\varepsilon_e}{\varepsilon} + 2C - \frac{5}{6} - \frac{7}{64} + \frac{63}{8} \int_0^\infty \ln z \varphi_2^{(5)}(z) dz \right) \\
&\quad + O\left(\frac{\varepsilon^3}{\varepsilon_e^3}\right) = 1 - \frac{4\pi}{15} \frac{\varepsilon}{\varepsilon_e} + \frac{64}{21} \frac{\varepsilon^2}{\varepsilon_e^2} \left(\ln \frac{\varepsilon_e}{\varepsilon} - 2.040 \right) + O\left(\frac{\varepsilon^3}{\varepsilon_e^3}\right). \quad (\text{A.13})
\end{aligned}$$

Here the first two terms agree with Eq.(3.9) in [10].

B Appendix

Here we calculate the function J_v in Eq.(A.1) defined in Eq.(A.3). In the last expression one can expand the function $l_1(z, t)$ up to terms $\propto z^4$. We find

$$\begin{aligned}
 J_{v1} &\simeq \left(\frac{\varepsilon}{\varepsilon_e}\right)^2 \int_0^1 (1-x)^2 dx \int_0^\infty dt e^{-t} \frac{t^4}{4!} \left(a_1^{(4)}(0)(\ln t + C) + b_1^{(4)}(0) \right) \\
 &= \frac{1}{3} \left(\frac{\varepsilon}{\varepsilon_e}\right)^2 \left[(\psi(5) - \psi(1)) a_1^{(4)}(0) + b_1^{(4)}(0) \right] \\
 &= \frac{1}{3} \left(\frac{\varepsilon}{\varepsilon_e}\right)^2 \left[\frac{25}{12} a_1^{(4)}(0) + b_1^{(4)}(0) \right], \tag{B.1}
 \end{aligned}$$

where the function $\psi(x)$ is defined in Eq.(2.2). The integral with the function $l_2(z, t)$ is

$$J_{v2} = -4\text{Re} \int_0^1 (1-x) dx \int_0^\infty dt e^{-t} [a_2(z)(\ln t + C) + b_2(z)]. \tag{B.2}$$

The term with $b_2(z)$ can be calculated as the term J_{m2} in the Appendix A. Substituting $\varphi_2(z) \rightarrow b_2(z)$ we have

$$\begin{aligned}
 J_{v2}^{(b)} &= -2b_2(0) - 2\pi \frac{\varepsilon}{\varepsilon_e} b_2^{(2)}(0) \\
 &\quad - 12 \left(\frac{\varepsilon}{\varepsilon_e}\right)^2 \left[\left(\ln \frac{\varepsilon_e}{\varepsilon} + 2C - \frac{5}{6} \right) b_2^{(4)}(0) - 2 \int_0^\infty \ln z b_2^{(5)}(z) dz \right]. \tag{B.3}
 \end{aligned}$$

Now we turn over to the term with $a_2(z)$ in Eq.(B.2)

$$\begin{aligned}
 T &= \text{Re} \int_0^1 (1-x) dx \int_0^\infty dt e^{-t} a_2(z)(\ln t + C) \\
 &= \text{Re} \int_0^1 (1-x) dx \int_0^\infty dt e^{-t} (a_2(z) - a_2(0)) (\ln t + C). \tag{B.4}
 \end{aligned}$$

As in previous Appendix we divide the integral over x into two: $0 \leq x \leq x_0$ and $x_0 \leq x \leq 1$. We have (see Eqs.(B.1) and (A.7))

$$\begin{aligned}
 T_1 &= \text{Re} \int_{x_0}^1 (1-x) dx \int_0^\infty dt e^{-t} (a_2(z) - a_2(0)) (\ln t + C) \\
 &= -\frac{1}{|\beta|^4} \int_{x_0}^1 \frac{(1-x)^3}{x^2} dx \frac{25}{12} a_1^{(4)}(0) \simeq -\frac{25}{12|\beta|^4} \left(\frac{1}{y_0^2} + 6 \ln y_0 + \frac{5}{2} \right) a_2^{(4)}(0). \tag{B.5}
 \end{aligned}$$

In the second integral we perform twice integration by parts over t and pass to the variable y . We find

$$\begin{aligned} T_2 &= \frac{2}{|\beta|^2} \text{Im} \int_0^{y_0} \frac{dy}{y(1+y^2)^3} \int_0^\infty dt e^{-t} \left[a_2^{(2)}(z)(\ln t + C) + d(z) \right] \\ d(z) &= \frac{2}{z} a_2^{(1)}(z) - \frac{1}{z^2} (a_2(z) - a_2(0)). \end{aligned} \quad (\text{B.6})$$

We proceed as above (see Eq.(A.8)) and expand $(1+y^2)^3 \simeq 1 - 3y^2$, then

$$\begin{aligned} T_{21} &= \frac{2}{|\beta|^2} \text{Im} \int_0^{\beta y_0} \frac{dy}{y} \int_0^\infty dt e^{-t} \left[a_2^{(2)}\left(\frac{t}{y}\right) (\ln t + C) + d\left(\frac{t}{y}\right) \right] \\ &= \frac{1}{|\beta|^2} \frac{1}{i} \int_{\beta^* y_0}^{\beta y_0} \frac{dy}{y} \int_0^\infty dt e^{-t} \left[a_2^{(2)}\left(\frac{t}{y}\right) (\ln t + C) + d\left(\frac{t}{y}\right) \right] \\ &\simeq \frac{1}{|\beta|^2} \left[\frac{\pi}{2} d(0) + \frac{1}{|\beta|^2 y_0^2} \left(\frac{3}{2} a_2^{(4)}(0) + d^{(2)}(0) \right) \right] \\ &= \frac{1}{|\beta|^2} \left[\frac{3\pi}{4} a_2^{(2)}(0) + \frac{25}{12|\beta|^2 y_0^2} a_2^{(4)}(0) \right]. \end{aligned} \quad (\text{B.7})$$

and

$$\begin{aligned} T_{22} &= -\frac{6}{|\beta|^2} \text{Im} \int_0^{y_0} y dy \int_0^\infty dt e^{-t} \left[a_2^{(2)}(z) (\ln t + C) + d\left(\frac{t}{y}\right) \right] \\ &= -\frac{6}{|\beta|^2} \text{Im} \int_0^{y_0} y dy \int_0^\infty dt e^{-t} \left[(a_2^{(2)}(z) - a_2^{(2)}(0)) (\ln t + C) + d\left(\frac{t}{y}\right) \right]. \end{aligned} \quad (\text{B.8})$$

Integrating by parts over t two times we obtain

$$\begin{aligned} T_{22} &= \frac{6}{|\beta|^4} \text{Re} \int_0^{\beta y_0} \frac{dy}{y} \int_0^\infty dt e^{-t} \left[a_2^{(4)}\left(\frac{t}{y}\right) (\ln t + C) + g\left(\frac{t}{y}\right) \right] \\ g(z) &= \frac{2}{z} a_2^{(3)}(z) - \frac{1}{z^2} (a_2^{(2)}(z) - a_2^{(2)}(0)) + d^{(2)}(z). \end{aligned} \quad (\text{B.9})$$

The term with the function $g(z)$ can be calculated in the same manner as in Eq.(A.10) ($\varphi_2^{(4)}(z) \rightarrow g(z)$)

$$T_{22}^{(g)} = \frac{6}{|\beta|^4} \text{Re} \left[(\ln(\beta y_0) + C) g(0) - \int_0^\infty \ln z g'(z) dz \right] \quad (\text{B.10})$$

Now we have to calculate the integral containing $\ln t + C$

$$\begin{aligned}
T_{22}^{(l)} &= \frac{6}{|\beta|^4} \operatorname{Re} \int_0^{\beta y_0} \frac{dy}{y} \int_0^\infty e^{-t} a_2^{(4)} \left(\frac{t}{y} \right) (\ln t + C) dt \quad (\text{B.11}) \\
&\simeq \frac{6}{|\beta|^4} \int_0^\infty \ln y dy \int_0^\infty z e^{-zy} (\ln y + \ln z + C) a_2^{(5)}(z) dz \\
&= \frac{6}{|\beta|^4} \int_0^\infty dy \int_0^\infty (\ln y - \ln z) (\ln y + C) e^{-y} a_2^{(5)}(z) dz \\
&= -\frac{6}{|\beta|^4} a_2^{(4)}(0) \int_0^\infty e^{-y} (\ln^2 y - C^2) dy = -\frac{\pi^2}{|\beta|^4} a_2^{(4)}(0)
\end{aligned}$$

Taking into account that $g(0) = \frac{25}{12} a_2^{(4)}(0)$ and summing T_1 Eq.(B.5), T_{21} Eq.(B.7), $T_{22}^{(g)}$ Eq.(B.10), $T_{22}^{(l)}$ Eq.(B.11) we find for T Eq.(B.4)

$$\begin{aligned}
T &= \frac{3\pi}{4} \frac{\varepsilon}{\varepsilon_e} a_2^{(2)}(0) \quad (\text{B.12}) \\
&\quad + \frac{\varepsilon^2}{\varepsilon_e^2} \left[\frac{25}{4} \left(\ln \frac{\varepsilon_e}{\varepsilon} + 2C - \frac{5}{6} - \frac{4}{25} \pi^2 \right) a_2^{(4)}(0) - 6 \int_0^\infty \ln z g'(z) dz \right]; \\
g(z) &= \frac{4}{z} a_2^{(3)}(z) - \frac{1}{z^2} (6a_2^{(2)}(z) - a_2^{(2)}(0)) + \frac{8}{z^3} a_2^{(1)}(z) - \frac{6}{z^4} (a_2(z) - a_2(0)).
\end{aligned}$$

Substituting the obtained expression for T Eq.(B.12) and $J_{v_2}^{(b)}$ Eq.(B.3) into J_{v_2} , adding to J_{v_1} and taking into account that

$$\begin{aligned}
a_1^{(4)}(0) &= \frac{1}{30}, \quad b_1^{(4)}(0) = \frac{1223}{450}, \quad a_2^{(2)}(0) = -\frac{4}{15}, \quad a_2^{(4)}(0) = \frac{16}{21}, \\
b_2(0) &= -\frac{1}{18}, \quad b_2^{(2)}(0) = -\frac{1}{225}, \quad b_2^{(4)}(0) = \frac{106}{735}, \quad (\text{B.13})
\end{aligned}$$

and numerical value of integrals

$$\int_0^\infty \ln z b_2^{(5)}(z) dz \simeq 0.250, \quad \int_0^\infty \ln z g'(z) dz \simeq 0.967, \quad (\text{B.14})$$

we obtain

$$J_v = \frac{1}{9} + \frac{182\pi}{225} \frac{\varepsilon}{\varepsilon_e} - \frac{15272}{735} \frac{\varepsilon^2}{\varepsilon_e^2} \left(\ln \frac{\varepsilon_e}{\varepsilon} - 2.577 \right) + O\left(\frac{\varepsilon^3}{\varepsilon_e^3} \right). \quad (\text{B.15})$$

C Appendix

Here we consider the asymptotic behavior of the total probability of radiation W in the region $\varepsilon \ll \varepsilon_e$ to within terms $\sim \varepsilon/\varepsilon_e$ inclusively. According with Eqs.(A.1)-(A.4) we present the total probability in form

$$W = \frac{1}{L_{rad}^0} \left(W_m + \frac{1}{L_1} W_v \right), \quad (C.1)$$

where

$$\begin{aligned} W_m &= 2\text{Re} \int_0^1 \frac{dx}{x} \int_0^\infty dt e^{-t} [x^2 \varphi_1(z) - 2(1-x)\varphi_2(z)]; \\ W_v &= -\text{Re} \int_0^1 \frac{dx}{x} \int_0^\infty dt e^{-t} [x^2 l_1(z, t) + 4(1-x)l_2(z, t)]. \end{aligned} \quad (C.2)$$

The function entering in Eq.(C.2) are defined in Eqs.(A.3)-(A.4).

The term with $\varphi_1(z)$ in the integrand of W_m to within mentioned accuracy is

$$W_{m1} = 2 \int_0^1 x dx \int_0^\infty e^{-t} dt \varphi_1(0). \quad (C.3)$$

In the integral with $\varphi_2(z)$ we divide the integral over x into two

$$W_{m2}^{(1)} = -4\text{Re} \int_{x_0}^1 \frac{dx}{x} (1-x) \int_0^\infty dt e^{-t} \varphi_2(z) \simeq 4(\ln x_0 + 1)\varphi_2(0), \quad (C.4)$$

and

$$\begin{aligned} W_{m2}^{(2)} &= -8\text{Re} \int_0^{y_0} \frac{dy}{y(1+y^2)^2} \int_0^\infty dt e^{-t} \varphi_2(z) \\ &\simeq -8\text{Re} \int_0^{\beta y_0} \frac{dy}{y} \int_0^\infty dt e^{-t} \varphi_2\left(\frac{t}{y}\right) \\ &\quad + 16\text{Re} \int_0^{y_0} y dy \int_0^\infty dt e^{-t} \varphi_2\left(\frac{t}{\beta y}\right) = W_{m21}^{(2)} + W_{m22}^{(2)} \end{aligned} \quad (C.5)$$

Integration of $W_{m21}^{(2)}$ coincides with calculation in Eq.(A.10) with substitution $\varphi_2^{(4)}\left(\frac{t}{y}\right) \rightarrow \varphi_2\left(\frac{t}{y}\right)$. We have

$$\begin{aligned} W_{m21}^{(2)} &= -8\text{Re} \int_0^{\beta y_0} \frac{dy}{y} \int_0^\infty dt e^{-t} \varphi_2\left(\frac{t}{y}\right) \\ &\simeq -4 \left[\left(\ln \frac{\varepsilon_e}{\varepsilon} + 2C + \ln y_0^2 \right) \varphi_2(0) - 2 \int_0^\infty \ln z \varphi_2^{(1)}(z) dz \right] \end{aligned} \quad (C.6)$$

Integrating by parts twice over t in integral $W_{m22}^{(2)}$ Eq.(C.5) we find (see Eq.(A.9))

$$W_{m22}^{(2)} = 16 \frac{\varepsilon}{\varepsilon_e} \text{Im} \int_0^{\beta y_0} \frac{dy}{y} \int_0^\infty dt e^{-t} \varphi_2^{(2)} \left(\frac{t}{y} \right) \simeq 4\pi \frac{\varepsilon}{\varepsilon_e} \varphi_2^{(2)}(0). \quad (\text{C.7})$$

Summing $W_{m2}^{(1)}$ Eq.(C.4), $W_{m21}^{(2)}$ Eq.(C.6) and $W_{m22}^{(2)}$ Eq.(C.7) and taking into account Eq.(A.12)) we have

$$\begin{aligned} W_m &= -4\varphi_2(0) \left[\ln \frac{\varepsilon_e}{\varepsilon} + 2C - 1 - \frac{\varphi_1(0)}{4\varphi_2(0)} - \frac{2}{\varphi_2(0)} \int_0^\infty \ln z \varphi_2^{(1)}(z) dz \right] + \\ 4\pi \frac{\varepsilon}{\varepsilon_e} \varphi_2^{(2)}(0) &= \frac{4}{3} \left[\ln \frac{\varepsilon_e}{\varepsilon} + 2C - \frac{5}{8} + 12 \int_0^\infty \ln z \left(\frac{1}{z^3} - \frac{\cosh z}{\sinh^3 z} \right) dz \right] + \frac{8\pi}{15} \frac{\varepsilon}{\varepsilon_e} \\ &= \frac{4}{3} \left(\ln \frac{\varepsilon_e}{\varepsilon} + 1.959 \right) + \frac{8\pi}{15} \frac{\varepsilon}{\varepsilon_e}. \end{aligned} \quad (\text{C.8})$$

The first term in Eq.(C.8)) is similar to Eq.(2.21) in [9].

We will consider now the probability W_v in Eq.(C.2)). The term with $l_1(z, t)$ gives no contribution into two first terms of expansion over $\frac{\varepsilon}{\varepsilon_e}$.

$$W_{v2} = -4\text{Re} \int_0^1 (1-x) \frac{dx}{x} \int_0^\infty dt e^{-t} [a_2(z)(\ln t + C) + b_2(z)] = W_{v2}^{(a)} + W_{v2}^{(b)}. \quad (\text{C.9})$$

The calculation of the term $W_{v2}^{(b)}$ (with $b_2(z)$) is analogous to the calculation above (with substitution $\varphi_2(z) \rightarrow b_2(z)$)

$$W_{v2}^{(b)} = -4b_2(0) \left(\ln \frac{\varepsilon_e}{\varepsilon} + 2C - 1 \right) + 8 \int_0^\infty \ln z b_2^{(1)}(z) dz + 4\pi \frac{\varepsilon}{\varepsilon_e} b_2^{(2)}(0). \quad (\text{C.10})$$

For calculation of term $W_{v2}^{(a)}$ (with $a_2(z)$) it is convenient to use the variable y

$$W_{v2}^{(a)} = W_{v21}^{(a)} + W_{v22}^{(a)}; \quad (\text{C.11})$$

$$W_{v21}^{(a)} = -8 \int_0^\infty \frac{dy}{y} \int_0^\infty dt e^{-t} \left(a_2 \left(\frac{t}{y} \right) - a_2 \left(\frac{1}{y} \right) \right) (\ln t + C),$$

$$W_{v22}^{(a)} = 8\text{Re} \int_0^\infty \frac{dy}{y} \left(1 - \frac{1}{(1+y^2)^2} \right) \int_0^\infty dt e^{-t} (a_2(z) - a_2(0)) (\ln t + C).$$

The integral over y in $W_{v21}^{(a)}$ is the integral Frullani

$$\int_0^\infty \left(a_2 \left(\frac{t}{y} \right) - a_2 \left(\frac{1}{y} \right) \right) \frac{dy}{y} = -a_2(0) \ln t. \quad (\text{C.12})$$

Using the last result in Eq.(B.11)) we have

$$W_{v21}^{(a)} = \frac{4}{3}\pi^2 a_2(0). \quad (\text{C.13})$$

In the expression for $W_{v22}^{(a)}$ we divide the integration interval into two parts:

$$\begin{aligned} W_{v22}^{(a1)} &= 8\text{Re} \int_{y_0}^{\infty} \frac{dy}{y} \left(1 - \frac{1}{(1+y^2)^2} \right) \\ &\quad \times \int_0^{\infty} e^{-t} \left(a_2^{(2)}(0) \frac{t^2}{2! \beta^2 y^2} + \dots \right) (\ln t + C) dt = 0, \\ W_{v22}^{(a2)} &\simeq 16\text{Re} \int_0^{y_0} y dy \int_0^{\infty} e^{-t} (a_2(z) - a_2(0)) (\ln t + C) dt \\ &= \frac{16}{|\beta|^2} \text{Im} \int_0^{\beta y_0} \frac{dy}{y} \int_0^{\infty} e^{-t} \left[a_2^{(2)} \left(\frac{t}{y} \right) (\ln t + C) + d \left(\frac{t}{y} \right) \right] dt. \end{aligned} \quad (\text{C.14})$$

The function $d(z)$ is defined in Eq.(B.6), the further calculation can be done as in Eq.(B.7)

$$W_{v22}^{(a2)} \simeq 4\pi \frac{\varepsilon}{\varepsilon_e} d(0) = 6\pi \frac{\varepsilon}{\varepsilon_e} a_2^{(2)}(0). \quad (\text{C.15})$$

Summing $W_{v2}^{(b)}$ Eq.(C.10), $W_{v21}^{(a)}$ Eq.(C.13) and $W_{v21}^{(a)}$ Eq.(C.15) we obtain the following expression for W_v Eq.(C.1)

$$\begin{aligned} W_v &= -4b_2(0) \left(\ln \frac{\varepsilon_e}{\varepsilon} + 2C - 1 \right) + 8 \int_0^{\infty} \ln z b_2^{(1)}(z) dz \\ &\quad + \frac{4\pi^2}{3} a_2(0) + 2\pi \frac{\varepsilon}{\varepsilon_e} (3a_2^{(2)}(0) + 2b_2^{(2)}(0)). \end{aligned} \quad (\text{C.16})$$

Taking into account that

$$\int_0^{\infty} \ln z b_2^{(1)}(z) dz = 0.0431$$

and using Eq.(B.13) we obtain $(a_2(0) = 1/3)$

$$W_v = \frac{2}{9} \ln \frac{\varepsilon_e}{\varepsilon} + 4.7655 - \frac{364\pi}{225} \frac{\varepsilon}{\varepsilon_e} + O \left(\frac{\varepsilon^2}{\varepsilon_e^2} \right). \quad (\text{C.17})$$

D Appendix

Here we discuss a structure of corrections to the total intensity of radiation at $\varepsilon \ll \varepsilon_e$ which is not included in the expansion over $1/L_c$ contained in Eqs.(2.1)- (2.4). We will show that these corrections, which are contained terms $\propto 1/L_1^2$ and higher degrees of $1/L_1$, are beginning from terms of the order $\varepsilon/\varepsilon_e$. We will use the general expression Eq.(2.12) of [7] before decomposition over $1/L$:

$$\frac{dI}{dx} = \frac{2\alpha m^2 x}{1-x} \text{Im} \langle 0 | r_1 (G^{-1} - G_0^{-1}) + r_2 \mathbf{p} (G^{-1} - G_0^{-1}) \mathbf{p} | 0 \rangle, \quad (\text{D.1})$$

where

$$G = \mathbf{p}^2 + 1 - iV(\boldsymbol{\rho}), \quad G_0 = \mathbf{p}^2 + 1, \quad V(\boldsymbol{\rho}) = \frac{\varepsilon}{4\varepsilon_e} \frac{1-x}{x} \boldsymbol{\rho}^2 \left(L_1 - \ln \frac{\boldsymbol{\rho}^2}{4} - 2C \right), \quad (\text{D.2})$$

here the quantities r_1, r_2 are defined in Eq.(2.2) and ε_e and L_1 are defined in Eq.(2.4). In the region $1 \geq x \gg \varepsilon/\varepsilon_e$ we expand the combination entering in Eq.(D.1) over degrees of V

$$G^{-1} - G_0^{-1} = G_0^{-1} iV G_0^{-1} + G_0^{-1} iV G_0^{-1} iV G_0^{-1} + G_0^{-1} iV G_0^{-1} iV G_0^{-1} iV G_0^{-1} + \dots \quad (\text{D.3})$$

Substituting Eq.(D.3) into Eq.(D.1) one can verify that the terms of the order $1/L_1^2$ and higher degrees of $1/L_1$ are appeared starting from the third term of decomposition Eq.(D.3). Their relative contribution is of the order $\varepsilon^2/\varepsilon_e^2$. Taking into account that

$$G_0^{-1} = i \int_0^\infty e^{-it} e^{-i\mathbf{p}^2 t} dt, \quad \langle \boldsymbol{\rho}_1 | e^{-i\mathbf{p}^2 t} | \boldsymbol{\rho}_2 \rangle = \frac{1}{4\pi i t} e^{\frac{i}{4t} (\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2)^2}, \quad (\text{D.4})$$

we obtain

$$\langle \boldsymbol{\rho}_1 | G_0^{-1} | \boldsymbol{\rho}_2 \rangle = \frac{1}{4\pi} \int_0^\infty \exp \left[-it + \frac{i}{4t} (\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2)^2 \right] \frac{dt}{t} = \frac{1}{2\pi} K_0(|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2|), \quad (\text{D.5})$$

where $K_0(z)$ is the modified Bessel function (Mac-Donald's function). Substituting this result into Eq.(D.3) and then into Eq.(D.1) we get

$$\begin{aligned} \frac{dI}{dx} &= \frac{\alpha m^2 x}{\pi(1-x)} \int_0^\infty [r_1 K_0^2(\varrho) + r_2 K_1^2(\varrho)] V(\varrho) \varrho d\varrho \quad (\text{D.6}) \\ &= \frac{\alpha m^2}{4\pi} \frac{\varepsilon}{\varepsilon_e} \left[\frac{r_1 + 2r_2}{3} L_1 + \frac{r_2 - r_1}{9} \right], \quad I^{(0)} = \frac{\alpha m^2}{4\pi} \frac{\varepsilon}{\varepsilon_e} \left(L_1 + \frac{1}{9} \right). \end{aligned}$$

Obtained here result for $I^{(0)}$ agrees with Bethe-Maximon formula.

In the region $x \sim \varepsilon/\varepsilon_e$ the potential $V \sim 1$ and decomposition Eq.(D.3) is inapplicable. However in this region

$$dI \sim \alpha m^2 \frac{xdx}{1-x} \sim \alpha m^2 \left(\frac{\varepsilon}{\varepsilon_e} \right)^2, \quad (\text{D.7})$$

and the relative contribution of this region $\sim \varepsilon/\varepsilon_e$.

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Variation of radiation length due to LPM effect

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