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THE STABILITY OF MULTIPOLE LONGITUDINAL OSCILLATIONS OF MULTIBUNCH BEAMS IN STORAGE RINGS WITH THE ACCOUNT OF BEAM COUPLING WITH THE ENVIRONMENT

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Contents

1 Introduction

This paper is devoted to the development of an apparatus for the study of the multipole synchrotron oscillations stability. We follow here a method developed in [1], where this problem was considered on two models: a model of macroparticles and a continuum model. The model of macroparticles does not allow to take into account the real particles distribution and the spread of synchrotron frequencies along the bunch. The second model gives more correct, but more complicated results, using the Vlasov equation, leading to the integral equation for a distribution function. This equation and the stability conditions for a distribution function of one bunch were considered in [1]. The stability conditions were obtained using the Nyquist criterion. But the application of this criterion in the case of the asymmetrical multibunch beam is rather difficult, if not impossible.

We develop here a method for calculating the growth rates of the multipole longitudinal oscillations of the arbitrary multibunch beam interacting with the RF structure cavities, under the same restrictions as in [1]:

- sinusoidal oscillations in the absence of excitation;
- small multipole perturbations due to the interaction (compared with the undisturbed distribution);
- the interaction only with the cavity modes with the wavelength greater than the bunch length;
- the amplitude dependence of the synchrotron frequency is taken into account in approach of small amplitudes, keeping only zero and first order terms of this dependence.

We assume also that the undisturbed distribution functions of all bunches are identical (in particular, all bunches have the same length), but their currents can be different. We have got the matrix equation, which contains the eigen values under the sign of integral. When the assumptions made above are valid, we can transform this equation to the linear eigen value problem (where the eigen value is a complex frequency shift if we neglect the amplitude dependence of the frequency, or a rather smooth integral function of the frequency shift in the other case, which can be used to find the shift itself for all unstable modes).

2 Development of a method for a multibunch beam

2.1 Vlasov equation

We start, following to [1], from the linearized Vlasov equation, subjected to the Laplace transform (assuming that the multipole oscillations are small as compared with the undisturbed distribution function):

$$sF(\psi, J, s) - \tilde{f}(\psi, J, 0) + \Omega\frac{\partial F}{\partial \psi} + L[\dot{J}]\frac{\partial f_0}{\partial J} = 0. \tag{1}$$

We use the next denotations: s – the Laplace variable; Ω – the synchrotron frequency; J and ψ – the action and synchrotron phase variables; f_0 – the undisturbed distribution function, not depending on the time in the rotating reference frame; $\tilde{f}(\psi, J, t)$ – the disturbance of the distribution function; $F(\psi, J, s)$ – its Laplace transform: $F(\psi, J, s) = L[\tilde{f}(\psi, J, t)]$.

A forcing term in (1) $L[\dot{J}]$ is determined from the equation

$$\dot{J} = -\frac{\partial H}{\partial \psi},$$

$$H = -e\int E_{z0}dz,$$

E_{z0} is the longitudinal component of the electric field on the equilibrium orbit in the reference system rotating with the equilibrium particle.

2.2 Separating distribution functions for all bunches

A distribution function in eq. (1) describes the whole multibunch beam. For our purposes, we rewrite the whole beam distribution function as a sum of separate distribution functions for all n_0 bunches in the beam:

$$F = \sum_{l=1}^{n_0} F^l.$$

As each bunch oscillates in it's own separatrix, we can further use the space orthogonality of the distribution functions of different bunches.

Thus, instead of one equation (1) we get a system of equations for bunches distribution functions:

$$sF^l(\psi, J, s) - \tilde{f}^l(\psi, J, 0) + \Omega\frac{\partial F^l}{\partial\psi} + L[\dot{J}]\frac{\partial f_0^l}{\partial J} = 0, l = 1, ..., n_0. \qquad (2)$$

Note that in $L[\dot{J}]$ the currents of all bunches are summarized, therefore (2) does not split into n_0 independent equations.

Following to [1], we can calculate the forcing term for the particle in the l-th separatrix with the longitudinal coordinate

$$z = \theta_l R + z_0(J)sin(\psi) = \frac{2\pi l}{n_0}R + z_0(J)sin(\psi) \qquad (3)$$

as

$$E_{z0}(z) = -\sum_{k,m} e^{imz/R} E_{zkm} Z_k(s - im\omega_0) I_{km}(s),$$

hence,

$$H(z) = e\sum_{k,m}\frac{R}{im}e^{imz/R}E_{zkm}Z_k(s - im\omega_0)I_{km}(s) =$$

$$= e\sum_{kmn} e^{in\psi}A_{mn}(J)\frac{R}{im}e^{im\theta_l}E_{zkm}Z_k(s - im\omega_0)I_{km}(s),$$

and

$$L[\dot{J}_l] = -\frac{\partial H}{\partial\psi} = -eR\sum_{kmn} e^{in\psi}A_{mn}(J)\frac{n}{m}e^{im\theta_l}E_{zkm}Z_k(s-im\omega_0)I_{km}(s), \qquad (4)$$

where

$$e^{ikm_l/R} = e^{im\theta_l}\sum_n e^{in\psi}A_{mn}(J),$$

$$I_{km}(s) = ecN\int E_{zk,-m}F(\psi, J, s)e^{-imz_j/R}d\psi dJ,$$

$$I_{km}(s) = \sum_{j=1}^{n_0} I_{km}^j(s) = \sum_{j=1}^{n_0} ecN_j\int E_{zk,-m}\sum_q e^{iq\psi'}F_q^j(J', s)e^{-imz_j/R}d\psi' dJ' =$$

$$= \sum_{j=1}^{n_0} ecN_j E_{zk,-m}\sum_q 2\pi\int F_q^j(J', s)A_{mq}^*(J')dJ' e^{-im\theta_j}.$$

The distribution functions can be spread into the Fourier series over the synchrotron phase ψ:

$$F(\psi, J, s) = \sum_n e^{in\psi} F_n(J, s), \quad \tilde{f}_0(\psi, J) = \tilde{f}(\psi, J, 0) = \sum_n e^{in\psi} f_{0n}(J). \quad (5)$$

Substituting (4) and (5) into (2), we finally get a system of integral equations for n-th harmonic of the multipole oscillations:

$$(s + in\Omega) F_n^l(J, s) -$$

$$- \frac{\partial f_0^l}{\partial J} \sum_{km} A_{mn}(J) \frac{n}{m} e^{im\theta_l} E_{zkm} Z_k(s - im\omega_0) I_{km}(s) = f_{0n}^l(J),$$

or

$$F_n^l(J, s) - eI_0 \sum_{j=1}^{n_0} \sum_q \int K_{qn}^{lj}(J, J', s) F_q^j(J', s) dJ' = \frac{f_{0n}^l(J)}{(s + in\Omega)}, \quad (6)$$

where

$$K_{qn}^{lj}(J, J', s) =$$

$$= \frac{\partial f_0^l}{\partial J} \frac{I_j/I_0}{(s + in\Omega)} \sum_m \frac{n}{m} Z_m(s - im\omega_0) A_{mn}(J) A_{mq}^*(J') e^{im(\theta_l - \theta_j)}, \quad (7)$$

$$Z_m(s - im\omega_0) = \sum_k Z_k(s - im\omega_0) |E_{zkm}|^2.$$

For small amplitudes of the synchrotron oscillations, we can consider them approximately as sinusoidal, thus the coefficients $A_{mn}(J)$ are

$$A_{nm}(J) = J_n \left(\frac{m}{R} \sqrt{\frac{2J}{M\Omega}} \right). \quad (8)$$

Near the resonance we can keep in (6) in a sum over q only the resonant term with $q = n$:

$$F_n^l(J, s) - \lambda \sum_{j=1}^{n_0} \int K_{nn}^{lj}(J, J', s) F_n^j(J', s) dJ' = \frac{f_{0n}^l(J)}{(s + in\Omega)}, \quad (9)$$

$$l = 1, ..., n_0, \quad \lambda = eI_0.$$

We have got a system of integral equations (9) for n_0 distribution functions $F_n^l(J, s)$. Further we shall drop the multipole number n - the index of F_n^l and K_{nn}^{lj}, keeping it only at $f_{0n}^l(J)$.

2.3 System of equations for counterrotating electron and positron beams

The system of equations obtained in previous section for multibunch electron beam can be generalized on the case of counterrotating electron-positron beams as it was made in [2] for the model of macroparticles: substitution of positron bunches with equivalent counterrotating electron bunches which pass the resonant cavity at the same time moment as the primary positron bunch. If the cavity is placed at the angular distance θ_0 from one of the points of beams interaction, the longitudinal coordinates (3) for particles in k-th equivalent bunch are

$$z = (\theta_k - 2\theta_0)R + z_0(J)sin(\psi) = (\frac{2\pi k}{n_0} - 2\theta_0)R + z_0(J)sin(\psi). \qquad (10)$$

Thus, the final system of equations for n_0 electron and n_0 positron bunches is the same system (9), but for $l = 1, ..., n_0, n_0+1, ..., 2n_0$ and with account of θ_0 in terms of (7) describing interaction of counterrotating beams:

$$\theta_l^{el,pos} - \theta_j^{el,pos} = \frac{2\pi(l-j)}{n_0},$$

$$\theta_l^{el,pos} - \theta_j^{pos,el} = \frac{2\pi(l-j)}{n_0} \pm 2\theta_0.$$

The upper indexes *el,pos* refer to electron and positron bunches correspondingly.

2.4 The number of eigen modes

Remember that in the model of macroparticles, the number of eigen modes of bunches oscillations is equal to the number of bunches, i.e. in the simplest case of one bunch there exists only one mode. But, if we consider the kernel of our integral equation, we shall see that it consists of a number of terms, to which, in general case, should be equal the number of independent solutions.

Now, we shall solve this contradiction.

Let's consider the degenerated kernel (7) for the simplest case of one bunch ($n_0 = 1$, $l = j = 1$). We want to show that in this case only one eigen function and its eigen value exists in reality. Dropping the subscripts l and j, we have

$$F_n(J,s) - \lambda \int K_{nn}(J,J',s)F_n(J',s)dJ' = 0, \qquad (11)$$

$$K_{nn}(J,J',s) = \frac{\frac{\partial f_0}{\partial J}}{(s+in\Omega)} \sum_{m=-N}^{N} \frac{n}{m} Z_m(s-im\omega_0) J_n(m\alpha\sqrt{J}) J_n(m\alpha\sqrt{J'}),$$

$$\alpha = \sqrt{\frac{2}{M\Omega R^2}}.$$

Here we consider the homogeneous equation in order to find the eigen values and eigen functions. The number of terms is limited by the maximum harmonic number N (due to finite frequency band of the impedance, for example). Replacing (near the resonance) s with $-in\Omega$ in Z_m and denoting $Z_m^{\pm} = Z_m(\mp in\Omega - im\omega_0)$, we have

$$K_{nn}(J,J') == \frac{\frac{\partial f_0}{\partial J}}{(s+in\Omega)} \sum_{m=1}^{N} \frac{n}{m} (Z_m^{+} - (Z_m^{-})^{*}) J_n(m\alpha\sqrt{J}) J_n(m\alpha\sqrt{J'}). \quad (12)$$

For a degenerated kernel with finite number of terms the solution can be searched in the form

$$F_n(J,s) = \frac{\partial f_0}{\partial J} \frac{1}{(s+in\Omega)} \times \sum_{m=1}^{N} B_m J_n(m\alpha\sqrt{J}). \quad (13)$$

Using orthogonality of Bessel functions $J_n(m\alpha x)$ for different m, we can write the matrix equation equivalent to the eq. (11):

$$B_m - \lambda \frac{n}{m} (Z_m^{+} - (Z_m^{-})^{*}) \sum_{m'=1}^{N} G_{m,m'} B_{m'} = 0, \quad (14)$$

$$G_{m,m'} = \int \frac{\partial f_0}{\partial J} \frac{1}{(s+in\Omega(J))} J_n(m\alpha\sqrt{J}) J_n(m'\alpha\sqrt{J}) dJ.$$

For a Gauss distribution

$$f_0(J) = \frac{1}{2\pi J_0} e^{-J/J_0}, \quad (15)$$

neglecting for a moment the amplitude dependence $\Omega(J)$, we can write, following to the formula (2.12.39(3)) of [3]:

$$\int_0^{\infty} x e^{-ax^2} J_\nu(bx) J_\nu(cx) dx = \frac{1}{2a} e^{-(b^2+c^2)/4a} I_\nu(bc/2a),$$

with denotations

$$x^2 = J, \quad a = 1/J_0, \quad b = m\alpha, \quad c = m'\alpha,$$

$$G_{m,m'} = \frac{-1}{4\pi(s + in\Omega(0))} e^{-(m^2+(m')^2)\alpha^2 J_0/4} I_n(m'm\alpha^2 J_0/2).$$

The equation (14) can be rewritten as

$$\sum_{m'=1}^{N} M_{m,m'} B_{m'} - \Lambda B_m = 0,$$

$$M_{m,m'} = C_m e^{-(m^2+(m')^2)\kappa/2} I_n(m'm\kappa),$$

$$C_m = -\frac{1}{4\pi}\frac{n}{m}(Z_m^+ - (Z_m^-)^*),$$

$$\kappa = \alpha^2 J_0/2 = \frac{J_0}{R^2 M\Omega}, \quad \Lambda = (s + in\Omega(0))/\lambda = \frac{s + in\Omega(0)}{eI_0}.$$

Using expressions for J_0 ($J_0 = \sigma^2 M\Omega$) and for synchrotron frequency ($\Omega^2 = (eqV sin(\phi_s))/(2\pi R^2|M|)$), where q and V are the RF harmonic number and voltage amplitude and σ is the r.m.s. length of the bunch, we can write

$$\kappa = \frac{2\pi|J_0|\Omega}{eqV sin(\phi_s)} = \frac{\sigma^2}{R^2}.$$

In approach of small bunch length, if $N^2\kappa \ll 1$, one can use approximation

$$I_n(m'm\kappa) \approx \frac{1}{n!}(\frac{(m'm\kappa)}{2})^n. \qquad (16)$$

In this case all columns of the matrix $M_{mm'}$ are proportional to the first column. With the transformation matrix

$$\hat{V}^{-1} = \left\{ \begin{array}{ccccc} E & -\frac{M_{12}}{M_{11}} & -\frac{M_{13}}{M_{11}} & \ldots & -\frac{M_{1N}}{M_{11}} \\ 0 & E & 0 & \ldots & 0 \\ & & \ldots & & \\ 0 & & \ldots & & E \end{array} \right\},$$

we have

$$\hat{M}' = \hat{V}\hat{M}\hat{V}^{-1} = \left\{ \begin{array}{cc} M'_{11} & M'_{1k} \\ M'_{n1} & M'_{nk} \end{array} \right\}, k, n = 2, ..., N,$$

where

$$M'_{11} = \sum_{m=1}^{N} C_m e^{-m^2\kappa} \frac{I_n^2(m\kappa)}{I_n(\kappa)};$$

$$M'_{n1} = M_{n1};$$

$$M'_{nk} = M_{nk} - M_{n1}\frac{M_{1k}}{M_{11}};$$

$$M'_{1k} = M_{1k} - M_{11}\frac{M_{1k}}{M_{11}} + \sum_{l=2}^{N} \frac{M_{1l}}{M_{11}}(M_{lk} - M_{l1}\frac{M_{1k}}{M_{11}}).$$

In approach of small bunch length, only first column contains nonzero elements, thus, the characteristic equation is

$$\begin{vmatrix} M'_{11}-\Lambda & 0 & \dots & 0 \\ M'_{21} & -\Lambda & \dots & 0 \\ \dots & \dots & \dots & \dots \\ M'_{N1} & 0 & \dots & -\Lambda \end{vmatrix} = 0.$$

Its solutions are

$$\Lambda_1 = M'_{11}, \;\; \Lambda_k = 0, k = 2, ..., M.$$

Λ_1 corresponds to the solution analogous to the solution with the model of macroparticles and all other zero solutions correspond to $1/\Lambda_k = \infty$ and to the zero terms in solution of the equation (6) (for the case of one bunch).

From another hand, in approach of small bunch length, the kernel (12) does not contain a set of independent functions, but all terms are proportional to one function $(JJ')^{n/2}$. As a result, there should be in fact one term in (13) and we get the matrix equation of order 1, with one eigen value. And when the terms of the kernel (12) become different at the bunch length, only then the matrix equation of order N will have another nontrivial solutions.

Increasing the bunch length, so that approximation (16) becomes not sufficient, we get nonzero elements of $\hat{M}'$ in 2-nd, 3-rd, ..., N-th columns. Keeping the first nonzero term in power series of these elements, we find them to be of order

$$M_{lk} - M_{l1}\frac{M_{1k}}{M_{11}^2} \sim \frac{I_n(\kappa)I_n(lk\kappa) - I_n(l\kappa)I_n(k\kappa)}{I_n^2(\kappa)} \sim \frac{(1-k^2)(1-l^2)}{4(n+1)}\kappa^2.$$

As a result, all eigenvalues (zero and nonzero) have a correction of order $(\frac{N^2\kappa}{2})^2$.Thus, there arise additional modes with nonzero Λ, whereas the first solution Λ_1 corresponding to the model of macroparticles only slightly changes (at small bunch length).

Thus, we have shown, that at small bunch length, we get the solutions corresponding to the model of macroparticles and increasing the bunch length leads to arising additional modes.

Obviously, in general case of n_0 bunches in the beam the whole set of $n_0 \cdot N$ eigen values should contain n_0 nonzero eigen values and all others eigen values in approach of small amplitudes should be equal to zero.

In the next section, we will consider the case of multibunch beam.

2.5 Transformation of the system of integral equations into one integral equation.

A system of integral equations of kind (9)

$$F^l(x) - \lambda \sum_{j=1}^{n_0} \int K^{lj}(x,x')F^j(x')dx' = f^l(x),$$

$$l = 1, ..., n_0, \quad x, x' \in [a, b]$$

can be rewritten as one equation (see [4]):

$$F'(x) - \lambda \int K'(x,x')F'(x')dx' = f'(x), \tag{17}$$

$$x, x' \in [a, a + n\Delta], \quad \Delta = b - a,$$

$$F'(x) = F^l(x - (l-1)\Delta), \quad \text{when} \quad x \in [a + (l-1)\Delta, a + l\Delta];$$

$$K'(x,x') = K^{lj}(x - (l-1)\Delta, x' - (j-1)\Delta),$$

$$\text{when} \quad x \in [a + (l-1)\Delta, a + l\Delta], x' \in [a + (j-1)\Delta, a + j\Delta].$$

A solution of eq. (17) can be written [4] via a resolvent $\Gamma(x, x', \lambda, s)$:

$$F'(x) = f'(x) + \lambda \int \Gamma(x, x', \lambda, s) f'(x')dx'. \tag{18}$$

The poles of $\Gamma(x, x', \lambda, s)$ are the zeroes of its denominator, an integral function $D(\lambda, s)$, each pole $\lambda_k(s)$ corresponds to the addentum of the solution with the time dependence $exp(s_k t)$, where $\lambda_k(s_k) = eI_0$. Note that the real part of s_k defines the growth rate of this addentum and $\lambda_k(s_k) = eI_0$ defines its amplitude. Therefore, our purpose is to find the zeroes s_k of the function $D(\lambda, s)$ at $\lambda = eI_0$.

This function can be written as a power series of λ (see [4], p.87) with the coefficients depending on s :

$$D(\lambda, s) = 1 + \sum_{k=1}^{\infty} \frac{(-\lambda)^k}{k!} C_k(s), \tag{19}$$

$$C_k(s) = \int_a^{a+n_0\Delta} Det_k|K'(x_\alpha, x_\beta)| dx_1 ... dx_k, \quad \alpha, \beta = 1, ..., k,$$

i. e.

$$C_k(s) = \int\limits_a^{a+n_0\Delta} ... \int \begin{vmatrix} K'(x_1, x_1) & ... & K'(x_1, x_k) \\ ... & ... & ... \\ K'(x_k, x_1) & ... & K'(x_k, x_k) \end{vmatrix} dx_1 ... dx_k =$$

$$= \sum_{r_1=1}^{n_0} ... \sum_{r_k=1}^{n_0} \int_{a+(r_1-1)\Delta}^{a+r_1\Delta} dx_1 ... \int_{a+(r_k-1)\Delta}^{a+r_k\Delta} dx_k Det_k|K^(x_\alpha, x'_\beta)| =$$

$$= \sum_{\substack{r_1 = 1 \\ ... \\ r_k = 1}}^{n_0} \int\limits_a^b ... \int \begin{vmatrix} K^{r_1 r_1}(x_1, x_1) & ... & K^{r_1 r_k}(x_1, x_k) \\ ... & ... & ... \\ K^{r_k r_1}(x_k, x_1) & ... & K^{r_k r_k}(x_k, x_k) \end{vmatrix} dx_1 ... dx_k. \tag{20}$$

In [1], a single bunch beam case was considered. Further, we shall calculate $C_k(s)$ for the kernel (7) describing the multibunch case.

2.6 Derivation of the equation kernel in the approach of small bunch length

For small amplitudes of the synchrotron oscillations, (i. e. for the harmonics with the wavelength much more than the bunch length), then the Bessel functions in (8) can keep only first term of their power series:

$$J_n\left(\frac{m}{R}\sqrt{\frac{2J}{M\Omega}}\right) \approx \frac{(m)^n}{n!}\left(\frac{J}{2R^2 M\Omega}\right)^{n/2} = \frac{(m)^n}{n!}\left(\frac{J\kappa}{2J_0}\right)^{n/2}. \tag{21}$$

If this approximation is valid, then the determinant in (20), with account (8) and (21), can be written as

$$Det_k|K^{r_i r_j}(x_i, x_j)| = \left(\frac{n}{(n!)^2}\right)^k (\frac{\kappa}{2J_0})^{nk} \times$$

$$\times \prod_{i+1}^{k} \left(\frac{I_{r_i}}{I_0} \frac{\partial f_0^{r_i}/\partial J_i}{s+in\Omega(J_i)} \right) Det_k |(J_i J_j)^{n/2} \hat{Z}_{r_i r_j}|,$$

where

$$\hat{Z}_{lj} = \sum_m m^{2n-1} e^{im(\theta_l - \theta_j)} Z(s - im\omega_0)$$

is the matrix of impedances. (Note that if the impedance has a resonant form, the summation over m can be fulfilled with the Watson-Zommerfeld transformation (see App.1) and $\hat{Z}$ can be presented as a sum over cavity resonant modes.)

The factors $(J_i)^{n/2}$ and $(J_j)^{n/2}$ can be removed from the determinant, thus all the factors depending on the action, which must be integrated in (20), are removed from the determinant:

$$Det_k |K^{r_i r_j}(x_i, x_j)| = \left\{ \prod_{i+1}^{k} \left(\frac{I_{r_i}}{I_0} \frac{\partial f_0^{r_i}/\partial J_i}{s+in\Omega(J_i)} J_i^n \right) \right\} \times$$

$$\times \left(\frac{n}{(n!)^2} (\frac{\kappa}{2J_0})^n \right)^k Det_k |\hat{Z}_{r_i r_j}|.$$

Calculating $C_k(s)$, we must integrate only the term in first braces.

Denoting

$$N_l = \int \frac{I_l}{I_0} \frac{\partial f_0^l/\partial J}{s+in\Omega(J)} (J/J_0)^n dJ, \quad \hat{N}_{lk} = N_l \delta_{lk}, \tag{22}$$

$$A = \frac{n}{(n!)^2} (\frac{\kappa}{2})^n,$$

$$\hat{S} = \hat{Z}\hat{N},$$

we get:

$$C_k(s) = A^k \sum_{\substack{r_1 = 1 \\ \dots \\ r_k = 1}}^{n_0} Det_k |\hat{S}_{r_i r_j}| = A^k k! \sum_{1 \le r_1 < r2 < \dots < r_k \le k}^{n_0} Det_k |\hat{S}_{r_i r_j}|.$$

The determinants $Det_k |\hat{S}_{r_i r_j}|$ represent the main minors of the order k of the matrix $\hat{S}$ of the order n_0:

$$\hat{S} = \begin{vmatrix} S_{11} & \dots & S_{1n_0} \\ \dots & \dots & \dots \\ S_{n_0 1} & \dots & S_{n_0 n_0} \end{vmatrix}.$$

It can be easily shown [5] that the sum (19) with these coefficients being the sums of the main minors of one matrix $\hat{S}$, has a form of a characteristic equation of matrix $\hat{S}$:

$$D(\lambda, s) = Det|\hat{S} - \frac{1}{A\lambda}\hat{E}|.$$

In this approach we get automatically $C_k = 0$ for $k > n_0$.

We have not still used the assumption of distribution functions identity for all bunches. If we do not take into account a dependence of the synchrotron frequency on the amplitude in (22) , we can factor the denominator depending on s outside the integral sign and calculate the matrix elements in (22) for different shapes of bunches. But the dependence of the synchrotron frequency can influence significantly on the multipole oscillations stability. To analyze this influence, we use the mentioned above assumption, that the undisturbed distribution functions of all bunches are identical, thus we can deal with a single integral function of s, instead of n_0 different functions for all bunches.

If the undisturbed distribution functions of all bunches are equal, i. e. $f_0^l = f_0$ (see (19)), then, denoting

$$\hat{N}'_{ij} = \frac{I_i}{I_0}\delta_{ij},$$

$$g(s) = \left\{ \int \frac{\partial f_0/\partial J}{s + in\Omega(J)} (J/J_0)^n \, dJ \right\}^{-1} \left\{ \int \frac{\partial f_0}{\partial J} (J/J_0)^n \, dJ \right\}, \qquad (23)$$

$$A_0 = A\lambda \left\{ \int \frac{\partial f_0}{\partial J} (J/J_0)^n \, dJ \right\} = \frac{n}{n!} \left(\frac{\sigma^2}{2R^2} \right)^{n-1} \frac{\Omega I_0}{2qV sin(\phi_s)},$$

we can write the dispersion equation $D(\lambda, s) = 0$ as

$$Det|A_0 \hat{Z} \hat{N}' - g(s)\hat{E}| = 0. \qquad (24)$$

Note that usually we assume that the zeroes of this function s_k are close to $-in\Omega$, which we substitute in $\hat{Z}$ to simplify the equation.

2.7 Complex frequency shift calculation

We have got the dispersion equation (24) in a form resembling the equation gotten via the model of macroparticles for the dipole oscillations. A variable to be found from this equation, a Laplace variable s, is included into it in a non-linear way. We deal with the nonlinearity in the matrix of impedances in usual way, substituting the zero approach $s = -in\Omega$ into its elements. Our main interest is a function g(s), nonlinearly depending on the value to be found.

Denote as Δs the complex shift of the multipole oscillation frequency (multiplied by i) for particles in the center of bunch (at $J = 0$):

$$\Delta s = s + in\Omega(0).$$

In the case of small amplitudes of oscillations we can neglect the amplitude dependence of the synchrotron frequency in (23) and hence

$$\Delta s_i = g_i, \tag{25}$$

that corresponds to the model of macroparticles.

When it is necessary to take into account the amplitude dependence of the frequency

$$\Omega(J) = \Omega(0)(1 - \xi\frac{J}{J_0})$$

(J_0 determines the size of the bunch, the derivations for ξ are given in App.2, see eq. (31)), we see that a problem breakes down into two steps:

1) calculating the eigen values g_i of the equation

$$Det|A_0\hat{Z}\hat{N}' - g_i\hat{E}| = 0 \tag{26}$$

and

2) calculating the values s_i as $g^{-1}(g_i)$, using iteration technique, for example.

Note that the Laplace transform of all functions was made for $Re(s) > 0$. Hense, the expression for $g(s)$ and the dispersional equation are obtained also for $Re(s) > 0$. In order to deal with a function depending on only one parameter, we define

$$z = \frac{s + in\Omega(0)}{-in\Omega_0\xi J_0}$$

and, for gaussian distribution,

$$G(z) = \left\{\int_0^\infty \frac{e^{-x}x^n}{z+x}dx\right\}^{-1}\left\{\int_0^\infty e^{-x}x^n dx\right\} = n!\left\{\int_0^\infty \frac{e^{-x}x^n}{z+x}dx\right\}^{-1} =$$

$$= \frac{g(z\cdot(-in\Omega_0\xi J_0) + in\Omega_0)}{-in\Omega_0\xi J_0}.$$

G(z) is defined for $Im(z) > 0$, which corresponds to $Re(s) > 0$.

The function G(z) reflects the upper semiplane of the variable z, (i.e. the right semiplane of the variable s) on the region V_1 at the fig.1. The maps on fig.1 correspond to lines with $Re(z) = const$ and $Im(z) = const$, for cases with n=1,2,3. Note that $G(0) = n$ and for $|z| \gg 1$ $G(z) \approx z + n + 1$.

The function $g(s)$ ($G(z)$) is defined, analytical and has the inverse function for $Re(s) > 0$ ($Im(z) > 0$). The poles s_k in this semiplane give the exponential growth of the distribution function $exp(s_k t)$. Therefore, for all dangerous modes the growth rates can be found as $s_k = g^{-1}(g_k)$. With other words, if $G_k = \frac{g_k}{-in\Omega_0 \xi J_0} \in V_1$, then the mode is unstable and its growth rate can be found with help of inverse function.

In the code, the growth rates can be found in two ways:

1) in approach of macroparticles, i.e. supposing the frequency spread along the bunch much smaller than $\Delta s = s + in\Omega(0)$, in this case $\Delta s_k = g_k$, and

2) for all unstable modes - with the continuum model - the correction to g_k, taking into account the finite bunch length and the frequency spread on this length, can be found as $\Delta s_k = g^{-1}(g_k)$, for $G_k \in V_1$, $Re(\Delta s_k) > 0$.

Note that in the case of $|g_i| < |\xi\Omega(0)|$ (i.e. for small current I_0 or for big r.m.s.length of the bunch σ) the solution can differ essentially from the model of macroparticles. In particular, the unstable modes for which g_i are in the region V_2 (see fig.1) become stable with account of synchrotron frequencies spread.

One can consider the value of frequency spread $|\xi\Omega(0)|$ as a character boundary between two models (the model of macroparticles and the continuum modes): if $|g_i| \gg |\xi\Omega(0)|$, the model of macroparticles describes the system very well and the iterations taking into account nonzero bunch length give negligibly small corrections and are redundant; in opposite case, if $|g_i| \leq |\xi\Omega(0)|$, the frequency spread decreases essentially the growth rate (or even eliminates instability at all) and changes the synchrotron shift.

3 A code for longitudinal multipole oscillations instabilities simulations

A code was written for computation of the growth rates and shifts of the synchrotron frequency of multipole oscillations, in resonant approach, i.e. when the obtained solutions are close to corresponding harmonic of the synchrotron frequency.

The data necessary for calculation are:

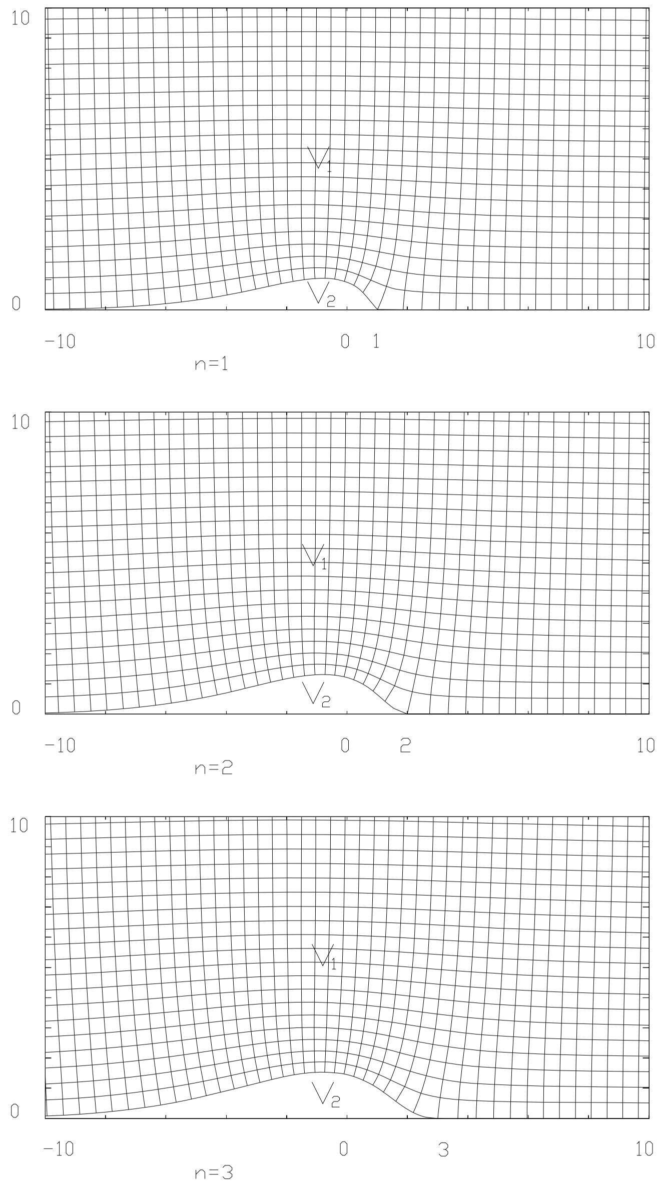

Figure 1: $G(z)$ for $Im(z) > 0$, n=1,2,3.

1. The parameters of a storage ring:

- the revolution frequency and the synchrotron frequency;
- the harmonic number;
- the accelerating voltage and synchronous RF phase;

2. The parameters of the beam:

- the average beam current;
- the particles energy;
- the number of bunches in the beam;
- the current of each bunch;
- the length of bunches;

3. RF system spectrum:

the resonant parameters (resonant frequencies, shunt resistances and quality factors) of all modes of all RF cavities to be taken into account or tabulated values of real and imaginary parts of the impedance in necesary frequency range, with sufficiently small frequency step;

4. The multipole number.

A code implies the possibilities:

1. to calculate the eigen values of the equation (26) analytically for a symmetrical beam, i. e. for a beam with equal distances between neighbour bunches and with equal charges of all bunches;

2. to solve numerically the eigenvalue problem for a beam with a gap;

3. to input different currents of all bunches (a most common case).

The growth rates and the shifts of the synchrotron frequency can be found from the eigen values of (26) in two ways:

1. without iterations, which corresponds to the model of macroparticles, when the amplitude dependence of the synchrotron frequency is neglected;

2. with iterations, taking into account the amplitude dependence of the synchrotron frequency in the approach of small amplitudes, when the spread of the synchrotron frequencies along the bunch is much less than the synchrotron frequency itself.

Note once more that the limits of the code reliability are

1) the considered impedance frequency range and the bunch length such that $\frac{\omega_{max}}{\omega_0}\sigma \leq 1$ and

2) calculated frequency shifts should be much less than the synchrotron frequency itself, in order to have the possibility to consider oscillations with different multipole numbers independently and to linearize the matrix equation, using not shifted synchrotron frequency at calculating impedances at side frequencies $m\omega_0 \pm n\Omega$. But if obtained frequency shifts for a some eigen mode appear to be too large, one can scan the solution in some range of synchrotron frequencies and choose the solution at that frequency, at which calculated frequency shift will be much less than the frequency itself.

Appendix 1

Applying the Watson-Sommerfeld trasformation to the kernel

The series

$$S(\theta) = \sum_{m=-\infty}^{+\infty} m^{2n-1} e^{im\theta} Z_m^+ \tag{27}$$

in the expression for the kernel (7) can be summed up in the case of resonant impedance having a form

$$Z(s) = \frac{\rho s'}{(s' - s_1)(s' - s_2)}, \tag{28}$$

$$s' = \frac{s}{\omega_r}, \;\; s_{1,2} = \pm i\nu_2 - \nu_1, \;\; \nu_1 = \frac{1}{2Q}, \;\; \nu_2 = \sqrt{1 - \nu_1^2}.$$

The series includes $Z(s - im\omega_0)$ for $s = -in\Omega$ (near the resonances of longitudinal oscillations). Thus, we should take the impedance at $s' = \frac{-im\omega_0 - in\Omega_z}{\omega_r} = -i\frac{m}{m_r} + \nu'$, where $m_r = \omega_r/\omega_0$, $\nu' = -in\Omega_z/\omega_r$.

In order to calculate the sum (27), we shall break the impedance (28) into 2 parts:

$$Z(s) = Z_1(s) + Z_2(s), \;\; S(\theta) = S_1(\theta) + S_2(\theta),$$

where $S_1(\theta)$ can be approximately substituted by the integral and $|Z_2(s)| \to 0$ at $|s| \to \infty$ so that the Watson-Zommerfeld transformation [6] could be applied to $S_2(\theta)$. Denoting $s_0 = s' - \nu'$ (s_0 is proportional to m), and $s'_{1,2} = s_{1,2} - \nu' = \pm i\nu_2 - \nu_1 - \nu'$, we have:

$$Z_1(s) = \frac{\rho s'/s_0}{s_1 - s_2} \sum_{k=0}^{2n-1} \left((\frac{s'_1}{s_0})^k - (\frac{s'_2}{s_0})^k \right),$$

$$Z_2(s) = \frac{\rho s'/s_0}{s_1 - s_2}((\frac{s_1'}{s_0})^{2n}\frac{1}{s-s_1} - (\frac{s_2'}{s_0})^{2n}\frac{1}{s-s_2}).$$

$S_1(\theta)$ can be calculated if we shall take into account the bunch angular length $\theta_0 = \sigma/R$:

$$S_1(\theta) = \sum_{m=-\infty}^{+\infty} m^{2n-1} e^{im\theta} Z^+_{1,m} e^{-m^2\theta_0^2} =$$

$$= \frac{\rho}{s_1 - s_2}\left[\sum_{k=1}^{n}(m_1^{2k-1} - m_2^{2k-1})S_{n-k}(\theta) + \sum_{k=2}^{n}(m_1^{2k-2} - m_2^{2k-2})n\nu_z S_{n-k}(\theta)\right],$$

where, for $\theta_0 \ll 1$,

$$S_k(\theta) = \sum_m m^{2k} e^{-m^2\theta_0^2} e^{im\theta} \approx (-\frac{d}{d(\theta_0^2)})^k(\frac{\sqrt{\pi}}{\theta_0}e^{-(\theta/2\theta_0)^2}) \approx$$

$$= \begin{cases} \frac{\sqrt{\pi}}{\theta_0}\frac{(2k-1)!!}{(2\theta_0^2)^k}, & \text{for } \theta = 0, \\ \frac{\sqrt{\pi}}{\theta_0}\left(\frac{\theta^2}{4\theta_0^2}\right)^k e^{-(\theta/2\theta_0)^2}, & \text{for } \theta \gg \theta_0. \end{cases}$$

$$m_{1,2} = im_r(s_{1,2} - \nu') = im_r(-\nu_1 \pm i\nu_2) - n\nu_z.$$

Now, let's calculate $S_2(\theta) = \sum_m f(m) = \sum_m m^{2n-1}e^{im\theta}Z^+_{2,m}$.

Considering m as continuous variable for $f(m)$, we can see that the only poles of the function f(m) are the poles $m_{1,2}$ of the impedance Z_2 in the upper semiplane ($m_{1,2}$ are defined above). Besides, the function f(z) in our case is such that the the integral over the upper semicircle C^+ $\int_{C^+} f(z)e^{-2\pi|Im(z)|}dz \to 0$ when the radius of this semicircle $R_C \to \infty$, if$-2\pi < \theta < 2\pi$.

According to Watson-Sommerfeld transformation [6], $S_2(\theta)$ can be calculated as

$$\sum_m f(m) = P.V.\int_{-\infty}^{+\infty} f(z)dz - \pi\sum_{1,2} Res_{1,2}(ctg(\pi m_{1,2}) + i),$$

with summing over both poles of all resonant modes of the impedance. In this expression $Res_{1,2}$ are the residuals of the function $f(z)$ in the points $m_{1,2}$:

$$Res_{1,2} = \frac{i\rho m_r}{2}(1 \pm i\frac{\nu_1}{\nu_2})m_{1,2}^{2n-1}e^{im_{1,2}\theta}$$

Finally, we have:

$$S_2(\theta) = -\frac{i\pi\rho m_r}{2}\sum_{1,2}(1 \pm i\frac{\nu_1}{\nu_2})m_{1,2}^{2n-1}e^{im_{1,2}\theta}(ctg(\pi m_{1,2}) - i\cdot sign(\theta)),$$

$$sign(\theta) = \begin{cases} 1, & 0 \le \theta < 2\pi, \\ -1, & -2\pi < \theta < 0. \end{cases}$$

The infinite sum S_2 is replaced by the finite sum with number of terms equal to double number of cavity resonant modes.

Appendix 2

The amplitude dependence of the frequency of synchrotron oscillations

Consider the equation of synchrotron oscillations:

$$\ddot{\phi} - \frac{\Omega^2}{\sin\phi_s}(\cos\phi - \cos\phi_s) = 0, \qquad (29)$$

ϕ is RF phase of the particle, ϕ_s is the RF phase of the synchronous particle, the accelerating voltage is $V\cos\phi_s$, Ω is the frequency of the synchrotron oscillations ($\Omega^2 = (eqV\,sin(\phi_s))/(2\pi R^2|M|)$).

For small deviations off the synchronous phase ϕ_s the equation (29) can be spread into Tailor series and, in first approach, describes harmonic oscillations with frequency Ω independent on the amplitude of oscillations. In order to find the amplitude dependence of the frequency, one should take into account higher terms of Tailor series.

We shall search the solution of the equation (29) in a form

$$\phi = \phi_s + \epsilon X(t),$$

where ϵ is a small parameter and $X(t)$ is a periodic function.

Spreading the equation (29) into Tailor series, we get the equation for $X(t)$:

$$\epsilon\ddot{X} + \frac{\Omega^2}{\sin\phi_s}(\sin\phi_s\epsilon X + \frac{1}{2!}\cos\phi_s(\epsilon X)^2 - \frac{1}{3!}\sin\phi_s(\epsilon X)^3 + ...) = 0.$$

or

$$\ddot{X} + \Omega^2 X = \epsilon\Omega^2(-\frac{1}{2!}\cot\phi_s X^2 + \frac{1}{3!}\epsilon X^3 + ...) \qquad (30)$$

At $\epsilon = 0$ the period of the solution $X_0(t)$ is $T_0 = \frac{2\pi}{\Omega}$. At $\epsilon \neq 0$ we search the solution $X(t)$ with a period

$$T = T_0(1 + a(\epsilon)) = T_0(1 + h_1\epsilon + h_2\epsilon^2 + h_3\epsilon^3 + ...),$$

in a form

$$X(t) = X_0(t) + \epsilon X_1(t) + \epsilon^2 X_2(t) + \epsilon^3 X_3(t) + ...$$

Turning to the problems "own time"

$$\tau = \frac{2\pi}{T_0(1 + a(\epsilon))}t = \frac{\Omega t}{1 + a(\epsilon)},$$

we can write

$$\ddot{X} = \frac{d^2X}{d\tau^2}\Omega^2(1 + \sum_{i=1}^{\infty}\epsilon^i h_i)^{-2}.$$

Substituting this expression into the eq. (30), and denoting the derivative on τ with a prime ($X'' = d^2X/d\tau^2$), we shall search the 2π-periodic solution of the equation:

$$(X_0'' + \sum_{i=1}^{\infty}\epsilon^i X_i'') + (X_0 + \sum_{i=1}^{\infty}\epsilon^i X_i)(1 + \sum_{i=1}^{\infty}\epsilon^i h_i)^2 =$$

$$= \epsilon(1 + \sum_{i=1}^{\infty}\epsilon^i h_i)^2(-\frac{\cot\phi_s}{2}(X_0 + \sum_{i=1}^{\infty}\epsilon^i X_i)^2 + \frac{\epsilon}{6}(X_0 + \sum_{i=1}^{\infty}\epsilon^i X_i)^3).$$

Comparing coefficients at different powers of ϵ, with account of supposition that the first harmonic of the solution is X_0 and $X_{1,2,...}$ contain only higher harmonics of the main period T, we can subsequently define the coefficients h_i and the functions X_i:

$$X_0(\tau) = Asin(\tau - \tau_0),$$

$$h_1 = 0,$$

$$X_1(\tau) = -\frac{A^2}{2}\frac{\cot\phi_s}{2}(1 + \frac{\cos 2(\tau - \tau_0)}{3}),$$

$$h_2 = \frac{A^2}{48}(5(\cot\phi_s)^2 + 3)...$$

The period of the solution with the firct amplitude-dependent correction is

$$T = T_0(1 + h_2\epsilon^2) = T_0(1 + \frac{(\epsilon A)^2}{48}(5(\cot\phi_s)^2 + 3).$$

Note that ϵA is the amplitude of the synchrotron oscillations in units of RF phase, hence

$$T = T_0(1 + (\frac{qz_0}{R})^2\frac{(5(\cot\phi_s)^2 + 3)}{48}),$$

where q is the RF harmonic number, R is the radius of the storage ring, z_0 is the amplitude of the longitudinal oscillations (in units of length):

$$z_0 = \sqrt{\frac{2J}{|M|\Omega}}.$$

For a Gaussian distribution (15) the r.m.s.length of the bunch is $\sigma^2 = \frac{J_0}{M\Omega}$. Thus, we can write (in a form convenient for integrals of section 2.6):

$$\Omega(J) = \Omega(0)(1 - \xi\frac{J}{J_0}), \;\; \xi = (\frac{q\sigma}{R})^2\frac{(5(\cot\phi_s)^2 + 3)}{24}. \qquad (31)$$

References

[1] *M.M. Karliner.* Coherent Instabilities of the Beam in Electron Storage Rings due to the Electromagnetic Interaction with the Surrounding Structure, Preprint INP 74-105, Novosibirsk, 1974 (in Russian).

[2] *M.M. Karliner, N.V. Mityanina, V.P. Yakovlev.* Longitudinal stability of colliding beams in e^+e^- storage rings with the account of beam coupling with the environment, Proceedings of the Third Advanced ICFA Beam Dynamics Workshop on Beam-Beam Effects in Circular Colliders. Novosibirsk, 1989, p.131-134.

[3] *A.P. Prudnikov et al.* Integrals and Series. Special Functions, Moscow, 1983 (in Russian).

[4] *S.G. Michlin.* Lectures on Linear Integral Equations, Moscow, 1959 (in Russian).

[5] *A.I. Maltsev.* Foundations of Linear Algebra, Moscow, 1970 (in Russian).

[6] *M. Month, R.F. Peierls.* Coupling Impedance structure above the tube cutoff frequency, NIM 137 (1976), p.299 (Appendix A).