Siberian Branch of Russian Academy of Science
BUDKER INSTITUTE OF NUCLEAR PHYSICS

N. Mityanina

THE STABILITY OF TRANSVERSE OSCILLATIONS OF MULTIBUNCH BEAMS IN STORAGE RINGS WITH THE ACCOUNT OF BEAM COUPLING WITH RF CAVITIES

Budker INP 99-45

NOVOSIBIRSK
1999

Contents

1 Introduction

In this paper, we consider the problem of transverse multibunch beam oscillations stability (including the case of counterrotating e^+e^- beams) when the beam interacts with the RF system of the storage ring, following to the method developed in [1], in approach of neglecting transverse current contribution in comparison with longitudinal one. We deal here with RF systems whose interaction with beam transverse oscillations can be described in terms of transverse impedance and of radially depending longitudinal impedance (for radial oscillations), in approach of separating longitudinal and radial variables.

Previously, we have considered the transverse oscillations instability for the multibunch beam interacting with resistive walls impedance [2]. Now, this problem, including the case of counterrotating electron and positron beams, is discussed for resonant impedance of RF cavities, taking into account their position relative the point of interaction of counterrotating bunches.

In approach of small bunch length in comparison with the wavelengthes of RF spectrum and comparatively small beam currents, using the model of macroparticles, the problem can be formulated as the eigenvalue problem for the matrix of order equal to the number of bunches in the beam (or in both beams in the case of counterrotating beams). The code MBI for multibunch beam instabilities calculation solves now this problem too.

2 Equations of motion

The equations of transverse motion in variables action-phase were obtained in [1] in a form (in linear approach):

$$
\begin{aligned}
\dot{\psi}_x &= \Omega_x(J_x, p_z) - e\tfrac{\partial x}{\partial J_x}(E_x - vB_y) = \Omega_x(J_x, p_z) + \Delta\Omega_x, \\
\dot{J}_x &= e\tfrac{\partial x}{\partial \psi_x}(E_x - vB_y) = 2J_x\sigma_x, \\
\dot{\psi}_y &= \Omega_y(J_y, p_z) - e\tfrac{\partial y}{\partial J_y}(E_y + vB_x + \tfrac{v}{eR}p_{zf})\Omega_y(J_y, p_z) + \Delta\Omega_y, \\
\dot{J}_y &= e\tfrac{\partial y}{\partial \psi_y}(E_y + vB_x + \tfrac{v}{eR}p_{zf}) = 2J_y\sigma_y, \\
\dot{p}_{zf} &= eE_{zf}.
\end{aligned}
\tag{1}
$$

Here we used the denotations:

$J_{x,y}$ and $\psi_{x,y}$ are action and phase for vertical and radial betatron oscillations;

$\Omega_{x,y}$ are frequencies of vertical and radial oscillations;

p_z is the longitudinal impulse of beam particles (in system of equilibrium particle, see [1]); p_{zf} is its fastly changing component;

$\sigma_{x,y}$ and $\Delta\Omega_{x,y}$ are the growth rate and the coherent shift for vertical and radial oscillations;

v is the particle velocity, R is the radius of the storage ring;

E and B are electric and magnetic fields.

The fields components in right hand sides of these equations are induced by harmonics of beam current $I_{km}(s)$ (see [1]):

$$\begin{aligned}
&E_x - vB_y = -\sum_{k,m} e^{i\frac{mz}{R}} L^{-1}\left[(sE_{kx,m} - \omega_0 R\tfrac{\partial E_{kz,m}}{\partial x}) \tfrac{Z_k(s-im\omega_0)}{s-im\omega_0} I_{km}(s)\right], \\
&E_y + vB_x = -\sum_{k,m} e^{i\frac{mz}{R}} L^{-1}\left[(sE_{ky,m} - \omega_0 R\tfrac{\partial E_{kz,m}}{\partial y}) \tfrac{Z_k(s-im\omega_0)}{s-im\omega_0} I_{km}(s)\right], \\
&E_{zf} = -\sum_{k,m} e^{i\frac{mz}{R}} E_{kz,m} L^{-1}\left[Z_k(s-im\omega_0) I_{km}(s)\right].
\end{aligned} \tag{2}$$

Here L^{-1} means reverse Laplace transform.

2.1 Vertical oscillations

Let consider at first vertical oscillations.

For small amplitudes of oscillations

$$x = \sqrt{\frac{2J_x}{m_s\Omega_x}} sin\psi_x = x_0 sin\psi_x \, , p_x = m_s\Omega_x \sqrt{\frac{2J_x}{m_s\Omega_x}} cos\psi_x \tag{3}$$

With these denotations, in order to consider one equation instead of pair of them, one can combine the equations of vertical motion as

$$x_0 e^{i\psi_x}(\sigma_x + i\Delta\Omega_x) = \hat{x}(\frac{\dot{J}_x}{2J_x} + i(\dot{\psi}_x - \Omega_x)) = \frac{e}{m_s\Omega_x}(E_x - vB_y), \tag{4}$$

where $\hat{x} = x_0 e^{i\psi_x} = \hat{x}_0 e^{i\Omega_x t}$, $\psi_x = \psi_{x0} + \Omega_x t$, $\hat{x}_0 = x_0 e^{i\psi_{x0}}$.

If we neglect longitudinal oscillations, the current density for the multi-bunch beam oscillating in vertical direction, is

$$\vec{j} = e\sum_{n=1}^{n_0} N_n \vec{v}_n \delta(x - x_n)\delta(y)\delta(z - R\theta_n), \tag{5}$$

where x_n describes the transverse oscillations of th n-th bunch: $x_n = x_{0n} sin(\psi_{0n} + \psi_x)$, and θ_n defines the longitudinal position of this bunch relative the first one: $\theta_n = 2\pi \frac{n-1}{n_0}, n = 1, ..., n_0$. We proposed here that the bunches occupy n_0 symmetrically placed along the orbit separatrices.

In the case of counterrotating beams, one can substitute the n_0 positron bunches with equivalent counterrotating electron bunches which pass the resonant cavity at the same time moment as the primary positron bunch (as it was made in [3] for longitudinal oscillations). If the RF cavity is placed at the angular distance θ_0 from one of the points of beams interaction, the longitudinal position of n-th equivalent bunch is

$$\theta_n = 2\pi \frac{n-1-n_0}{n_0} - 2\theta_0, n = n_0+1, ..., 2n_0.$$

The currents of equivalent electron bunches should be added to (5), with their values of θ_n.

The current harmonic for the current density (5) is

$$I_{km}(t) = \int \vec{E}_{k,-m} j_z(x,y,z) e^{-i\frac{mz}{R}} dxdydz = \sum_n eN_n \vec{v}\vec{E}_{k,-m}(x_k) e^{-im\theta_n},$$

$$I_{km}(s) = \sum_n eN_n e^{-im\theta_n} L\left[\vec{v}\vec{E}_{k,-m}(x_k)\right] =$$

$$= \sum_n eN_n e^{-im\theta_n} L\left[\frac{p_{xn}}{m_s} E_{kx,-m} + v x_n \frac{\partial E_{kz,-m}}{\partial x}\right],$$

here we have dropped the constant term $E_{kz,-m}(0)$ in paraxial spread of longitudinal field because it will give zero addition after averaging over the time [1].

Expressing x_n and p_{xn} via variables action-phase (3) and substituting $I_{km}(s)$ into expressions for fields components (2), one can put them into the r.h.s. of equations of motion (1) and average them over the time.

Obtained equations contain terms with $\frac{\partial E_{kz,m}}{\partial x}$ and with $E_{kx,m}$. We propose now that RF system is such that near the axis

$$|\omega_0 R \frac{\partial E_{kz,m}}{\partial x}| \gg |\Omega_x E_{kx,m}|,$$

i.e. one can neglect transverse currents in comparison with dipole component of the longitudinal current. In this case we can denote the dipole longitudinal and transverse impedances of the cavity as

$$Z_m(im\omega_0, x) = (2\pi R)^2 \sum_k |E_{kz,m}(x)|^2 Z_k(im\omega_0) = Z(im\omega_0, x)$$

and

$$Z_t(-i\omega) = \frac{c}{\omega}\frac{Z(-i\omega, x)}{x^2}.$$

With these denotations, the equation of motion (4) for the k-th bunch becomes ($\hat{x}_k = \hat{x}_{0k}e^{i\Omega_x t}$, $\hat{x}_{0k} = x_{0k}e^{i\psi_{0k}}$ - its complex amplitude):

$$\hat{x}_{0k}(\sigma_x + i\Delta\Omega_x) = e^{-i\Omega_x t}\frac{e}{m_s\Omega_x}(E_x - vB_y) =$$

$$= -i\frac{e^2\omega_0^2}{m_s\Omega_x c(2\pi)^2}\sum_m\sum_n N_n e^{im(\theta_k-\theta_n)}e^{-i\Omega_x t}L^{-1}\left[Z_t(s - im\omega_0)L[x_n]\right] =$$

$$= -\frac{e^2\omega_0^2}{2m_s\Omega_x c(2\pi)^2}\sum_m\sum_n\left\{N_n e^{im(\theta_k-\theta_n)}e^{-\Omega_x t}\times\right.$$

$$\left.\times L^{-1}\left[Z_t(s - im\omega_0)(\frac{\hat{x}_{0n}}{s - i\Omega_x} - \frac{\hat{x}^*_{0n}}{s + i\Omega_x})\right]\right\}. \qquad (6)$$

Using shift in complex plane of Laplace transform and averaging over the time, we get

$$\hat{x}_{0k}(\sigma_x + i\Delta\Omega_x) = -\frac{\omega_0}{2V_s\nu_x(2\pi)^2}\sum_m\sum_n I_n e^{im(\theta_k-\theta_n)}Z_t(i\Omega_x - im\omega_0)\hat{x}_{0n},$$

$$I_n = ecN_n,\ V_s = m_s c^2/e,\ \nu_x = \Omega_x/\omega_0.$$

In matrix form, with denotations:

$$S_{kn} = \sum_m e^{im(\theta_k-\theta_n)}Z_t(i\Omega_x - im\omega_0), \qquad (7)$$

$$\lambda = \sigma_x + i\Delta\Omega_x,\ \ A = \frac{\omega_0 I_0}{2V_s\nu_x(2\pi)^2}\ \ N_{kn} = \delta_{kn}I_n/I_0,$$

one can write

$$\lambda\vec{X} = A\hat{S}\hat{N}\vec{X},$$

where $\vec{X}$ is a vector of complex amplitudes of bunches oscillations.

Note that this equation describes the case of counterrotating beams too, with correctly defined θ_k for positron beam.

In the case of several RF cavities placed at different distances from the point of interaction of bunches, S_{kn} should be summed over all cavities, with account of angular position of each cavity.

In the case of one electron beam, the same equation can be applied for transverse resistive instability analysis, with correctly defined transverse resistive impedance.

2.2 Radial oscillations

For radial oscillations, the equations are analogous, but due to presence of p_{zf}, there appears an additional term (see [1]) in r.h.s. of (6):

$$e^{-i\Omega_y t}\frac{e}{m_s\Omega_y}\frac{v}{eR}p_{zf}.$$

In approach of neglecting the transverse currents, the terms with $E_{ky,m}$ in expression for current harmonic can be dropped and the addition to r.h.s. of (6) contains only terms with factors $E_{kz,m}\frac{\partial E_{kz,m}}{\partial y}$. These factors are not equal to zero for unsymmetric cavities, for modes in which both E_z and its transverse (radial) derivative simultaneously are not equal to zero. It leads to the linear dependence of longitudinal impedance on transverse deviation:

$$Z(s,y) = Z(s,0) + Z'(s)y,$$

$$Z'(s) = (2\pi R)^2 \sum_k (E_{kz,-m}\frac{\partial E_{kz,m}}{\partial y} + E_{kz,m}\frac{\partial E_{kz,-m}}{\partial y})Z_k(s).$$

If the variables y and z can be separated, i.e. if $E_{kz} = F_1(y)F_2(z)$ (as in waveguides or cylindric cavities, for example), then

$$Z'(s) = (2\pi R)^2 \sum_k 2E_{kz,-m}\frac{\partial E_{kz,m}}{\partial y}Z_k(s).$$

In this case, the additional term is

$$e^{-i\Omega_y t}\frac{e}{m_s\Omega_y}\frac{v}{eR}p_{zf} = -\frac{e^2\omega_0^2}{2m_s\Omega_y c(2\pi)^2}\sum_m\sum_n \Big\{ N_n e^{im(\theta_k-\theta_n)}e^{-\Omega_y t}\times$$

$$\times L^{-1}\left[\frac{\omega_0 Z'(s-im\omega_0)}{s}(\frac{\hat{y}_n}{s-i\Omega_y} - \frac{\hat{y}_n^*}{s+i\Omega_y})\right]\Big\}.$$

After shift in complex plane of Laplace transform and averaging over the time, this expression becomes

$$-\frac{\omega_0}{2V_s\nu_y(2\pi)^2}\sum_m\sum_n I_n e^{im(\theta_k-\theta_n)}\frac{Z'(i\Omega_y - im\omega_0)}{i\nu_y}\hat{y}_n.$$

Thus, the matrix $\hat{S}$ for radial oscillations has a form

$$S_{kn} = \sum_m e^{im(\theta_k-\theta_n)}(Z_t(i\Omega_y - im\omega_0) - \frac{Z'(i\Omega_y - im\omega_0)}{i\nu_y}). \qquad (8)$$

3 Transformation to the symmetric modes

It can be useful to use obtained matrix equation after transformation to the symmetric modes of multibunch beam oscillations. In this case, if we consider one beam in a storage ring, the matrix $\hat{S}$ becomes diagonal:

$$V_{kn} = exp(-i\frac{2\pi}{n_0}(k-1)(n-1)); \quad (V^{-1})_{kn} = \frac{1}{n_0}exp(i\frac{2\pi}{n_0}(k-1)(n-1));$$

$$S' = VSV^{-1},$$

$$S'_{kn} = n_0\delta_{kn}\sum_p Z^-_{t,(pn_0+k)},$$

where $Z^-_{t,m} = Z_{t,m}(-i(m\omega_0 - \Omega_x))$. Thus, the growth rates of eigen modes of symmetric beam are

$$\sigma_k = \frac{\omega_0 I_0}{2V_s\nu_x(2\pi)^2}\sum_p Re(Z^-_{t,(pn_0+k)}).$$

In the case of counterrotating beams, one should transform them to symmetric modes separately and as a result, the matrix $\hat{S}$ consists of 4 diagonal matrices of order n_0:

$$\hat{S}' = \begin{pmatrix} \tilde{S}^{11} & \tilde{S}^{12} \\ \tilde{S}^{21} & \tilde{S}^{22} \end{pmatrix},$$

$$\tilde{S}^{11}_{kn} = \tilde{S}^{22}_{kn} = \delta_{kn}\sum_p Z^-_{t,(pn_0+k)},$$

$$\tilde{S}^{12}_{kn} = \delta_{kn}\sum_p Z^-_{t,(pn_0+k)}exp(2i(pn_0+k)\theta_0),$$

$$\tilde{S}^{21}_{kn} = \delta_{kn}\sum_p Z^-_{t,(pn_0+k)}exp(-2i(pn_0+k)\theta_0),$$

The eigen values can be defined separately for each pair of symmetric modes of counterrotating beams with the same symmetry.

4 The series summation with the Watson-Sommerfeld transformation

In the case of low losses, when the longitudinal impedance can be presented as

$$Z(-i\omega, x) = \frac{R_{sl}(x)}{1 - iQ(\frac{\omega}{\omega_r} - \frac{\omega_r}{\omega})},$$

the transverse impedance is

$$Z_t(-i\omega) = \frac{R_{st}c/\omega}{1 - iQ(\frac{\omega}{\omega_r} - \frac{\omega_r}{\omega})},$$

where $R_{st} = R_{sl}(x)/x^2 = Q\rho_t$ does not depend on x.

Note that here we assume R_s with account of transit factor and in fact namely the transit factor depends on transverse coordinates and can be differentiated with respect to them. For radial oscillations, one should change x for y in these expressions.

For this impedance, the elements of the matrix $\hat{S}$ can be calculated with help of Watson-Sommerfeld transformation [4]. With this transformation, the sum over all harmonics of the revolution frequency turns into the sum over the cavity resonant modes (note that both functions, $Z_t(s)$ and $Z'(s)$, allow to apply this transformation):

$$S_1(\theta) = \sum_{m=-\infty}^{+\infty} e^{im\theta} Z^-_{t,m} = \sum_{1,2} \frac{i\pi\rho_t c}{2\nu_2\omega_0}(\pm 1)e^{im_{1,2}\theta}(ctg(\pi m_{1,2}) - i\cdot sign(\theta));$$

$$S_2(\theta) = -\frac{1}{i\nu_y}\sum_{m=-\infty}^{+\infty} e^{im\theta} Z'^-_m =$$

$$= \sum_{1,2} \frac{\pi\rho' m_r}{2\nu_y}(1 \pm i\frac{\nu_1}{\nu_2})e^{im_{1,2}\theta}(ctg(\pi m_{1,2}) - i\cdot sign(\theta));$$

$$m_{1,2} = im_r(-\nu_1 \pm i\nu_2) + \nu_{x,y};\ \ m_r = \frac{\omega_r}{\omega_0};\ \ \rho' = \frac{\partial R_{sl}(y)}{Q\partial y};$$

$$sign(\theta) = \begin{cases} 1, & 0 \le \theta < 2\pi, \\ -1, & -2\pi < \theta < 0. \end{cases}$$

In order to apply the transformation, one should take $-2\pi < \theta < 2\pi$.

5 The features of the code for calculating transverse instabilities

The growth rates and cogerent shifts for eigen modes of multibunch beam (or counterrotating electron and positron beams) transverse oscillations, can be calculated with the code MBI, in approaches made above:

- small bunch length in comparison with wavelengthes of the RF spectrum (which anables to use the model of macroparticles);
- small amplitude of transverse oscillations (such that the linear approach of motion equations and of electric and magnetic fields can be applied);
- neglecting the transverse currents in comparison with dipole component of longitudinal one (which enables to simplify the RF spectrum representation).

The RF spectrum ($Z_t(s)$ and $Z'(s)$) can be presented as a set of parameters of resonant modes of the cavities or as a table of measured or calculated values of real and imaginary parts of the impedance in some frequency region.

Another parameters necessary for calculation are the revolution frequency and the frequency of transverse oscillations; and the parameters of the beams (the particles energy, the current of each bunch, the r.m.s. length of the bunches; the number of bunches in beams and their positions along the orbit). In the case of counterrotating beams, the angular positions of RF cavities relative one of the points of bunches interaction also should be inputted.

References

[1] *M.M. Karliner.* Coherent Instabilities of the Beam in Electron Storage Rings due to the Electromagnetic Interaction with the Surrounding Structure, Preprint INP 74-105, Novosibirsk, 1974 (in Russian).

[2] *M.M. Karliner, N.V. Mityanina, V.P. Yakovlev.* Multibunch resistive wall instabilities of an intense electron beam in storage rings, Preprint BudkerINP 92-52, Novosibirsk, 1992.

[3] *M.M. Karliner, N.V. Mityanina, V.P. Yakovlev.* Longitudinal stability of colliding beams in e^+e^- storage rings with the account of beam coupling with the environment, Proceedings of the Third Advanced ICFA Beam Dynamics Workshop on Beam-Beam Effects in Circular Colliders, Novosibirsk, 1989, p.131-134.

[4] *M. Month, R.F. Peierls.* Coupling impedance structure above the tube cutoff frequency, NIM 137 (1976), p.299 (Appendix A).