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ON THE GENERALIZED ENERGY PRINCIPLE FOR THE FLUTE PERTURBATIONS

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Abstract

Expression is derived for the energy of the flute perturbations in a plasma in which there exists a population of "hot" particles whose drift frequency around the magnetic axis greatly exceeds the frequency of the perturbation. It is shown that the contribution of the hot particles to the overall perturbation energy is of the kinetic type (i.e., scales as $\dot{\xi}^2$). This result considerably affects the conclusions on the stability of various systems containing hot particles.

The velocity of the drift motion in non-uniform magnetic fields scales linearly with particle energy. Therefore, if the system contains particles with high enough energy, their drift frequency Ω_d may exceed the characteristic frequency of the flute perturbations Γ. Then, to analyze the stability of the system, one cannot use the familiar Kruskal–Oberman energy principle [1] and must switch to its modified version [2, 3] that takes into account the condition

$$\Omega_d \gg \Gamma \tag{1}$$

(the so-called "generalized energy principle"). However, if one inserts into the expression for the energy variation W, presented in the papers [2, 3], the displacement vector $\boldsymbol{\xi}$ of the flute perturbation,

$$\boldsymbol{\xi} = [\nabla\chi, \mathbf{B}]/B^2, \tag{2}$$

with χ constant along the field line, then one finds that W, as given in [2, 3], becomes identically zero (see Appendix). In the present paper, we resolve this paradox and provide the effective expression for the energy of flute perturbations.

We consider purely electrostatic perturbations that are characterized by a constant value of the electrostatic potential φ along the field line (the latter assumption, identical to the one made in [2], implies the presence of a cold plasma component). To describe the bounce-averaged motion of a particle, we use the following coordinate system. First, we assume that at some (possibly, large) distance from the confinement region, the magnetic field is transformed to a uniform one, without violating the magnetic field in the confinement region. (Such a situation is really met in some mirror devices, that incorporate a solenoid with a uniform magnetic field, but one can use this procedure also as a conceptual one). Then, we mark every field line with the polar coordinates r, ψ of its intersection with some plane perpendicular to the uniform magnetic field (Fig.1). Instead of r, it is sometimes more convenient to use the magnetic flux Φ inside the cylindrical surface of the radius r : $\Phi = \pi r^2 B_0$, where B_0 is the uniform magnetic field. A drift surface can then be described by the equation $r = r(\psi)$ (or $\Phi = \Phi(\psi)$).

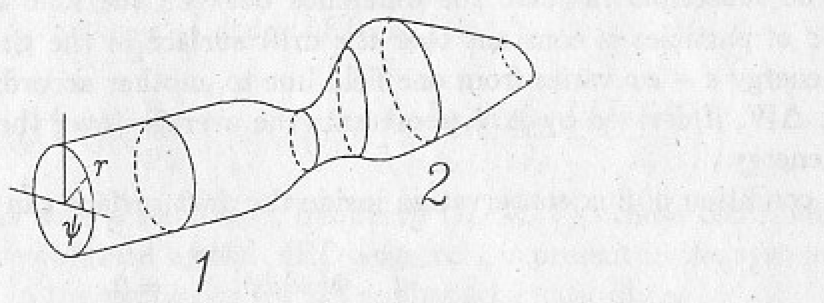

Fig. 1 Magnetic flux surface: *1*—solenoid region, *2*—confinement region.

At a given configuration of magnetic and electrostatic field, the drift surface for a particle with a total energy ε and magnetic moment μ is determined by the constancy of the longitudinal action $J(\varepsilon, \mu, \Phi, \psi) = \int v_\parallel dl$, with the integration carried out between the turning points, and

$$v_\parallel = (2/M)^{1/2}(\varepsilon - \mu B - e\varphi)^{1/2}. \tag{3}$$

If the condition (1) is satisfied, then, with the electrostatic potential varying, the drift surface adjusts itself to keep constant the magnetic flux inside the surface [4]. This occurs via variation of the particle energy.

To find the change W of the kinetic energy of the particles (just this quantity enters the energy principle for the perturbations with a scale-length much in excess of the Debye radius), we use the following approach. We consider some group of particles (of a total number ΔN)

that in the initial state have certain values of ε and μ, and that are filling a drift surface characterized with a certain value of J. When we slowly turn on the electrostatic potential of the perturbation, the drift surface deforms and the kinetic energy of the particles changes. If we find the change of the kinetic energy ΔW for this group, then, by summation over all the groups, we find the required quantity W.

The group ΔN is drifting along the contour $\Phi(\psi)$ determined by the instantaneous configuration of the electrostatic field and the instantaneous value of ε. The number of particles from this group $d\Delta N$, occupying the section of the contour of the arc length $d\psi$, can be presented in the form $d\Delta N = \nu d\psi$, where ν is the number of particles per unit arc length. The stationarity condition $\nu\dot{\psi} = const$ yields:

$$\nu = \frac{\Delta N \Omega_d}{2\pi\dot{\psi}}, \tag{4}$$

where $\dot{\psi}$ is the angular velocity of the bounce-averaged drift motion [5]:

$$\dot{\psi} = -\frac{2\pi Mc}{et_{\|}} J_\Phi, \tag{5}$$

$t_{\|} = MJ_\varepsilon$ is the transit time between the turning points, and Ω_d is the drift frequency,

$$\Omega_d = 2\pi\left[\int_0^{2\pi}\frac{d\psi}{\dot{\psi}}\right]^{-1}. \tag{6}$$

We use the notation $J_\varepsilon \equiv \partial J/\partial\varepsilon$, $J_\Phi \equiv \partial J/\partial\Phi$, etc., for the partial derivatives. The change of the kinetic energy of these particles is, obviously,

$$\Delta W = \left[\int_0^{2\pi}\nu(\varepsilon - e\varphi)d\psi\right]_{\text{initial}}^{\text{final}}, \tag{7}$$

where the subscripts indicate the difference between the final and initial state. The total energy ε of particles is constant over the drift surface in the time-scale of $\Omega_d{}^{-1}$, while their kinetic energy $\varepsilon - e\varphi$ varies from one field line to another according to variation of φ. In this respect, ΔW, if divided by ΔN, represents the average (over the drift surface) change of the kinetic energy.

The condition of flux conservation inside the drift surface can be written in the form:

$$\int_0^{2\pi}\Phi(\psi)d\psi\Big|_{\text{initial}}^{\text{final}} = 0, \tag{8}$$

where $\Phi(\psi)$ is a solution of the equation

$$J(\varepsilon,\mu,\Phi,\psi)|_{\text{initial}}^{\text{final}} = 0. \tag{9}$$

In principle, the equations (4)-(9) allow one to find the particle kinetic energy at arbitrarily large φ. However, we will consider only the case of small φ. The quantities of the first order in φ will be denoted with subscript "1", the second order corrections by subscript "2", etc.

In the linear approximation, equation (9) yields:

$$(\varepsilon_1 - e\varphi)J_\varepsilon + \Phi_1 J_\Phi = 0, \tag{10}$$

from which, taking into account relationships (4) and (7), we find that

$$\varepsilon_1 = \frac{\Omega_d}{2\pi}\int_0^{2\pi} e\varphi\frac{d\psi}{\dot{\psi}} \equiv e\langle\varphi\rangle, \tag{11}$$

where the drift average is defined as $\langle\ldots\rangle = \Omega_d/(2\pi)\int_0^{2\pi}(\ldots)d\psi/\dot{\psi}$.

The next order expansion of (9) gives:

$$J_\varepsilon(\varepsilon_2 - e\Phi_1\varphi_\Phi) + J_\Phi\Phi_2 + \frac{1}{2}J_{\varepsilon\varepsilon}(\varepsilon_1 - e\varphi)^2 + J_{\varepsilon\Phi}\Phi_1(\varepsilon_1 - e\varphi) + \frac{1}{2}J_{\Phi\Phi}\Phi_1{}^2 = 0. \tag{12}$$

The condition (8), when applied to Φ_2, yields:

$$\int_0^{2\pi}\frac{\varepsilon_2 - e\Phi_1\varphi_\Phi}{\dot{\psi}}d\psi = -\int_0^{2\pi}\frac{d\psi}{\dot{\psi}J_\varepsilon}\left(\frac{1}{2}J_{\varepsilon\varepsilon}(\varepsilon_1 - e\varphi)^2 + J_{\varepsilon\Phi}\Phi_1(\varepsilon_1 - e\varphi) + \frac{1}{2}J_{\Phi\Phi}\Phi_1{}^2\right), \tag{13}$$

while from (7) we find that

$$\Delta W = \frac{\Delta N\Omega_d}{2\pi}\int_0^{2\pi}\left[-\frac{\dot{\psi}_1}{\dot{\psi}}(\varepsilon_1 - e\varphi) + (\varepsilon_2 - e\Phi_1\varphi_\Phi)\right]\frac{d\psi}{\dot{\psi}}. \tag{14}$$

From equation (5) one obtains that

$$\frac{\dot{\psi}_1}{\dot{\psi}} = -\frac{J_\varepsilon}{J_\Phi}e\varphi_\Phi - \frac{J_\varepsilon}{J_\Phi}\left(\frac{J_{\Phi\Phi}}{J_\Phi} - \frac{2J_{\varepsilon\Phi}}{J_\varepsilon} + \frac{J_{\varepsilon\varepsilon}J_\Phi}{J_\varepsilon{}^2}\right)(\varepsilon_1 - e\varphi). \tag{15}$$

Now, using the relationships (5), (10), (11), (13) and (15), we can express ΔW in terms of φ:

$$\Delta W = -\frac{1}{2}e^2\Delta N\left\langle 2\frac{J_\varepsilon}{J_\Phi}(\varphi - \langle\varphi\rangle)\frac{\partial\varphi}{\partial\Phi} + (\varphi - \langle\varphi\rangle)^2\left(2\frac{J_{\varepsilon\Phi}}{J_\Phi} - \frac{J_{\varepsilon\varepsilon}}{J_\varepsilon} - \frac{J_\varepsilon J_{\Phi\Phi}}{J_\Phi^2}\right)\right\rangle. \tag{16}$$

To perform the summation over the plasma particles, we introduce the distribution function $F(\varepsilon,\mu,J)$, normalized according to the relationship $\Delta N = F(\varepsilon,\mu,J)\Delta\varepsilon\Delta\mu\Delta J$. Then, the energy W of the perturbation acquires its final form:

$$W = -\frac{1}{2}e^2\int d\varepsilon d\mu dJ F(\varepsilon,\mu,J)\times$$

$$\times\left\langle 2\frac{J_\varepsilon}{J_\Phi}(\varphi - \langle\varphi\rangle)\frac{\partial\varphi}{\partial\Phi} + (\varphi - \langle\varphi\rangle)^2\left(2\frac{J_{\varepsilon\Phi}}{J_\Phi} - \frac{J_{\varepsilon\varepsilon}}{J_\varepsilon} - \frac{J_\varepsilon J_{\Phi\Phi}}{J_\Phi^2}\right)\right\rangle. \tag{17}$$

If a population of particles with a small drift frequency ($\Omega_d \ll \Gamma$), whose contribution to the perturbation energy can be obtained by the MHD approach, is present in the system, then there appear two more terms in the expression for W, scaling schematically as

$$\mathcal{M}\dot{\xi}^2 + \mathcal{R}\xi^2. \tag{18}$$

The first term represents the kinetic energy of perturbations and the second one describes their potential energy (determined by the field line curvature). The expression (17) for W scales as φ^2. Since the displacement ξ of the fluxtube filled with a cold plasma is determined by the formula (2), with $\chi = (1/c)\int\varphi dt$, we see that if (17) is expressed in terms of ξ, it scales as the kinetic energy of the perturbations ($\sim\dot{\xi}^2$), giving contribution to $\mathcal{M}$, not to $\mathcal{R}$, in the expression (18). Therefore the presence of fast drifting particles manifests itself in the changing of the "inertia" of perturbations, not of their "rigidity". Just these observations explain a somewhat paradoxical result of paper [2] – cancellation of the energy of the flute perturbation. The reason, as we see, is that the authors of paper [2] retained only terms proportional to ξ^2, while the contribution of the fast particles to the energy of the perturbation has a different structure (it is proportional to $\varphi^2 \sim \dot{\xi}^2$).

To illustrate possible effect of non-MHD response of the fast particles, we consider a single non-paraxial mirror of length L (with a plasma occupying a volume of the order of L^3). Let plasma consist of a thermal population with temperature T and density n, and a hot population with temperature T_* and density $n_* < n$; let also the pressure of the hot component exceed that of the cold one: $n_*T_* > nT$. For the usually most dangerous mode of a global displacement one can evaluate the plasma kinetic energy (per unit volume) as

$$\left(\frac{n_*T_*}{\Omega_d{}^2L^2} + nM\right)\dot{\xi}^2,$$

where ξ is a (small) plasma displacement. The first term here represents a contribution of the fast particles. The potential energy is just $nT(\xi/L)^2$, as fast particles do not contribute to it. If the drift frequency of the fast particles Ω_d is not too high,

$$\Omega_d < \frac{1}{L}\left(\frac{n_*T_*}{nM}\right)^{1/2},$$

the inertia of the fast particles dominates. The estimate for the growth-rate Γ is then

$$\Gamma \sim \Omega_d\left(\frac{nT}{n_*T_*}\right)^{1/2}.$$

As $n_*T_* > nT$ the growth-rate is automatically less than the drift frequency, ensuring the applicability of our analysis. So, we see that, indeed, the "inertia" of the fast drifting particles can be dominant, despite their small density. Similar effects can play a significant role also for the systems containing several linked mirrors some of which have a population of hot particles (e.g., for tandem mirror systems [7, 8]). This part of the problem will be considered elsewhere.

Expression (17) can be considerably simplified for the widely used Yin–Yang configuration (see, e.g. [6]). An important feature of such a configuration is that, if we use the coordinate frame with the axis coinciding with the magnetic axis, then longitudinal invariant J, up to the terms linear in Φ, doesn't depend on the azimuthal angle ψ (see [6]):

$$J \cong J^{(0)}(\varepsilon,\mu) + \Phi J^{(1)}(\varepsilon,\mu).$$

The neglect of the higher order terms in Φ is justified in the paraxial region. In the framework of the paraxial approach, the plasma radial dimension should be small compared to the mirror-to-mirror distance. This means that the variation of φ in Φ has a small scale-length, and the dominant term in (17) is that containing the derivative $\partial\varphi/\partial\Phi$. This allows one to reduce the general expression (17) to a simplified form :

$$W = -\frac{1}{2\pi}e^2\int d\varepsilon d\mu dJ.\ F\ \frac{J_\varepsilon^{(0)}}{J^{(1)}}\int_0^{2\pi}(\varphi - \langle\varphi\rangle)\frac{\partial\varphi}{\partial\Phi}d\psi. \quad (19)$$

We have taken into account the independence of both $J^{(0)}$ and $J^{(1)}$ on ψ. Changing the set of the integration variables, and performing integration by parts, one can finally obtain:

$$W = \frac{1}{4\pi}e^2\int d\varepsilon d\mu d\Phi\ \frac{J_\varepsilon^{(0)}\left|J^{(1)}\right|}{J^{(1)}}\ \frac{\partial F}{\partial\Phi}\ \int_0^{2\pi}d\psi(\varphi - \langle\varphi\rangle)^2. \quad (20)$$

Since the derivative $\partial F/\partial\Phi$ defines the sign of the diamagnetic frequency, it becomes obvious from (5), (20) that the energy variation would be negative for those particles whose directions of the drift due to the field line curvature and the diamagnetic drift coincide, and would be positive in the opposite case.

Appendix

Van Dam-Rosenbluth-Lee energy principle in the case of the flute perturbations

We start with the introducing the Clebsh coordinates (α,θ) [9] with property

$$\mathbf{B} = \nabla\alpha\times\nabla\theta.$$

The α coordinate is chosen so that the contour surfaces of constant α form a nested series of topological cylinders, and it is normalized to enclose the magnetic flux $2\pi\alpha$ by any α surface. The θ coordinate is angle-like and of period 2π on each α surface. In the limit of zero β the magnetic field satisfies equation $\nabla\times\mathbf{B} = 0$, and hence it can be expressed as a gradient of some potential χ:

$$\mathbf{B} = \nabla\chi. \quad (21)$$

Vectors

$$(\nabla\alpha\ ,\nabla\theta\ ,\nabla\chi) \quad (22)$$

compose a covariant basis that we are going to deal with. We also define a contrvariant basis $(\mathbf{u},\mathbf{v},\boldsymbol{\tau})$, dual to (22), in such a way that

$$\mathbf{u} = \frac{\nabla\theta\times\nabla\chi}{B^2},$$

$$\mathbf{v} = -\frac{\nabla\alpha\times\nabla\chi}{B^2},$$

$$\boldsymbol{\tau} = \frac{\nabla\alpha\times\nabla\theta}{B^2} \equiv \frac{\mathbf{b}}{B}.$$

According to the energy principle [2], derived within the assumption of the fast particle drift (see condition (1)), the energy variation W consists of two terms:

$$W = W_f + W_k. \quad (23)$$

The first one, W_f, represents the local part of W and can be written in the Taylor–Hastie form [10] :

$$W_f = \frac{1}{2}\int d^3x\left[\sigma Q_\perp^2 + \zeta Q_\parallel^2 + \sigma j_\parallel\mathbf{b}\cdot(\boldsymbol{\xi}\times\mathbf{Q}) + q\boldsymbol{\xi}\cdot\nabla' p_\parallel\right.$$
$$\left. -(1/B)(2Q_\parallel + \boldsymbol{\xi}\cdot\nabla B)(\boldsymbol{\xi}\cdot\nabla' p_\perp)\right]. \quad (24)$$

Here $\boldsymbol{\xi}$ is the displacement vector, $\mathbf{Q}$ is the Eulerian magnetic field perturbation,

$$\mathbf{Q} = \nabla\times(\boldsymbol{\xi}\times\mathbf{B}),$$

the subscripts $\parallel,\perp$ refer to the parallel and perpendicular components with respect to the direction of the unperturbed magnetic field, and the coefficients σ and ζ,

$$\sigma = 1 - B^{-1}(\partial p_\parallel/\partial B), \qquad \zeta = 1 - (\partial^2 p_\parallel/\partial B^2),$$

are measures of stability against firehose and mirror anisotropy modes, respectively. Also, the following notations are introduced:

$$\nabla' = \nabla - (\nabla B)\partial/\partial B,$$

$$q = \mathbf{bb} : \nabla\boldsymbol{\xi}. \tag{25}$$

The kinetic contribution to the energy variation, W_k, originated from the fast particle drift, is given by

$$W_k = -\frac{1}{2}\int d\alpha d\theta d\mu dJ \left[\left(\frac{\partial F}{\partial \varepsilon}\right)_J \langle\langle H\rangle\rangle^2 + \left(\frac{\partial J}{\partial \varepsilon}\right)\left(\frac{\partial F}{\partial J}\right)_\varepsilon \langle H\rangle^2\right], \tag{26}$$

where

$$H = -mv_\parallel^2 q - \mu B(\nabla\cdot\boldsymbol{\xi} - q), \tag{27}$$

and $F = F(J(\alpha,\theta,\varepsilon,\mu),\varepsilon,\mu)$ is the equilibrium distribution function, depending only on the integrals of motion (see e.g. [11]). Single angle brackets in (26) describe the bounce average:

$$\langle\ldots\rangle = \left(\frac{\partial J}{\partial \varepsilon}\right)^{-1} \oint dl v_\parallel^{-1}(\ldots),$$

while double angle brackets denote the average both over bounce and drift motions:

$$\langle\langle\ldots\rangle\rangle = \left(\frac{\partial \tilde{\Phi}}{\partial \varepsilon}\right)_J^{-1} \oint d\theta \left(\frac{\partial \varepsilon}{\partial \alpha}\right)_J^{-1} \langle\ldots\rangle, \tag{28}$$

where $\tilde{\Phi} = \oint d\theta\alpha(\theta)$ is the flux adiabatic invariant ($\alpha = \alpha(\theta)$ defines the particle drift-surface, and integration is performing with J and ε being constant), $(\partial\tilde{\Phi}/\partial\varepsilon)_J$ is the precessional drift period, and $(\partial\varepsilon/\partial\alpha)_J = \langle d\beta/dt\rangle$ is the bounce-averaged rate of precession [4].

Now we turn to the calculation of the energy (23) for the flute-like perturbations, characterized by the following displacement vector $\boldsymbol{\xi}$:

$$\boldsymbol{\xi} = \mathbf{b}\xi_\parallel + \boldsymbol{\xi}_\perp, \tag{29}$$

$$\boldsymbol{\xi}_\perp = \frac{\mathbf{B}\times\nabla\eta}{B^2}, \tag{30}$$

with function η constant along the field line,

$$\eta = \eta(\alpha,\theta).$$

Since

$$\nabla\eta = \frac{\partial\eta}{\partial\alpha}\nabla\alpha + \frac{\partial\eta}{\partial\theta}\nabla\theta,$$

the perpendicular displacement (30) can be expressed as

$$\boldsymbol{\xi}_\perp = -\frac{\partial\eta}{\partial\theta}\mathbf{u} + \frac{\partial\eta}{\partial\alpha}\mathbf{v}. \tag{31}$$

In the curved coordinate system (α,θ,l) (with l as a coordinate along the field line, determined by $dl = d\chi/B$) one obtains:

$$\nabla\cdot\boldsymbol{\xi} = B\frac{\partial}{\partial l}\left(\frac{\xi_\parallel}{B}\right) + B^2\left[\frac{\partial}{\partial\theta}\left(\frac{1}{B^2}\frac{\partial\eta}{\partial\alpha}\right) - \frac{\partial}{\partial\alpha}\left(\frac{1}{B^2}\frac{\partial\eta}{\partial\theta}\right)\right] =$$

$$= B\frac{\partial}{\partial l}\left(\frac{\xi_\parallel}{B}\right) - \frac{2}{B}\left[\frac{\partial B}{\partial\theta}\frac{\partial\eta}{\partial\alpha} - \frac{\partial B}{\partial\alpha}\frac{\partial\eta}{\partial\theta}\right]. \tag{32}$$

As it flows from (25), (29), the expression for q yields:

$$q = \frac{\partial\xi_\parallel}{\partial l} - \boldsymbol{\xi}_\perp\cdot(\mathbf{b}\cdot\nabla)\mathbf{b}.$$

Accounting for $\nabla\times\mathbf{B} = 0$, and using the relationships

$$(\mathbf{b}\cdot\nabla)\mathbf{b} = -\mathbf{b}\times(\nabla\times\mathbf{b}),$$

$$\nabla\times\mathbf{b} = -\frac{1}{B}(\nabla B\times\mathbf{b}),$$

we come to

$$q = \frac{\partial\xi_\parallel}{\partial l} - \frac{1}{B}\left[\frac{\partial B}{\partial\theta}\frac{\partial\eta}{\partial\alpha} - \frac{\partial B}{\partial\alpha}\frac{\partial\eta}{\partial\theta}\right]. \tag{33}$$

After simple manipulations one can find from (27), (32), (33) that

$$\langle H\rangle = \left(\frac{\partial J}{\partial\varepsilon}\right)^{-1}\int\frac{dl}{v_\parallel}\,\xi_\parallel\frac{\partial}{\partial l}\left(\frac{Mv_\parallel^2}{2} + \mu B\right) +$$

$$+ \left(\frac{\partial J}{\partial\varepsilon}\right)^{-1}\int\frac{dl}{Bv_\parallel}\left(Mv_\parallel^2 + \mu B\right)\left[\frac{\partial B}{\partial\theta}\frac{\partial\eta}{\partial\alpha} - \frac{\partial B}{\partial\alpha}\frac{\partial\eta}{\partial\theta}\right]. \tag{34}$$

The contribution from the first integral vanishes, since the particle energy conserves along the field line.

As it was shown in [9],

$$\frac{\partial J}{\partial\alpha} = -\int\frac{dl}{Bv_\parallel}\left(Mv_\parallel^2 + \mu B\right)\frac{\partial B}{\partial\alpha}, \tag{35}$$

$$\frac{\partial J}{\partial\theta} = -\int\frac{dl}{Bv_\parallel}\left(Mv_\parallel^2 + \mu B\right)\frac{\partial B}{\partial\theta}. \tag{36}$$

Combining (35),(36) with (34), one can rewrite the expression for $\langle H\rangle$ as

$$\langle H\rangle = -\left(\frac{\partial J}{\partial\varepsilon}\right)^{-1}\frac{\partial J}{\partial\theta}\frac{\partial\eta}{\partial\alpha} + \left(\frac{\partial J}{\partial\varepsilon}\right)^{-1}\frac{\partial J}{\partial\alpha}\frac{\partial\eta}{\partial\theta}. \tag{37}$$

Now it is easy to perform the average of $\langle H\rangle$ over the drift motion. Inserting (37) into (28) and taking into account that

$$\left(\frac{\partial\alpha}{\partial\theta}\right)_J = -\frac{\partial J}{\partial\theta}\left(\frac{\partial J}{\partial\alpha}\right)^{-1}, \quad \left(\frac{\partial\varepsilon}{\partial\alpha}\right)_J = -\frac{\partial J}{\partial\alpha}\left(\frac{\partial J}{\partial\varepsilon}\right)^{-1},$$

we have:

$$\langle\langle H\rangle\rangle = \left(\frac{\partial\tilde{\Phi}}{\partial\varepsilon}\right)_J^{-1}\oint\left(d\theta\frac{\partial\eta}{\partial\theta} + d\alpha\frac{\partial\eta}{\partial\alpha}\right) = \left(\frac{\partial\tilde{\Phi}}{\partial\varepsilon}\right)_J^{-1}\oint d\eta = 0.$$

Finally, using

$$\left(\frac{\partial F}{\partial\alpha}\right)_\varepsilon = -\left(\frac{\partial J}{\partial\varepsilon}\right)\left(\frac{\partial\varepsilon}{\partial\alpha}\right)_J\left(\frac{\partial F}{\partial J}\right)_\varepsilon,$$

one obtains after the substitution of (37) into (26):

$$W_k = -\frac{1}{2}\int d\alpha d\theta d\mu d\varepsilon\left(\frac{\partial J}{\partial\alpha}\right)\left(\frac{\partial F}{\partial\alpha}\right)_\varepsilon\left(\left(\frac{\partial\eta}{\partial\theta}\right)^2 -\right.$$

$$\left. -2\left(\frac{\partial J}{\partial\theta}\right)\left(\frac{\partial J}{\partial\alpha}\right)^{-1}\left(\frac{\partial\eta}{\partial\alpha}\right)\left(\frac{\partial\eta}{\partial\theta}\right) + \left(\frac{\partial J}{\partial\alpha}\right)^{-2}\left(\frac{\partial J}{\partial\theta}\right)^2\left(\frac{\partial\eta}{\partial\alpha}\right)^2\right). \tag{38}$$

Here we have changed the set of the integration variables, and perform the integration over $d\varepsilon$ instead of dJ.

Now we turn to the calculation of the energy variation W_f. Since W_f does not depend on the parallel component $\xi_{\|}$ (see [10]), and besides that the displacement (29) does not perturb the magnetic field, $\mathbf{Q} = 0$, the expression (24) reduces to

$$W_f = -\frac{1}{2}\int d^3x \left(\frac{1}{B}(\boldsymbol{\xi}_\perp \cdot \nabla B)(\boldsymbol{\xi}_\perp \cdot \nabla' p_\perp) + \frac{1}{B}(\boldsymbol{\xi}_\perp \cdot \nabla' p_{\|})\left[\frac{\partial B}{\partial \theta}\frac{\partial \eta}{\partial \alpha} - \frac{\partial B}{\partial \alpha}\frac{\partial \eta}{\partial \theta}\right]\right). \tag{39}$$

The parallel and perpendicular pressure components, presented in (39), are given by

$$p_{\|} = \int \frac{d\varepsilon d\mu}{v_{\|}} B \;\; M v_{\|}^2 F,$$

$$p_\perp = \int \frac{d\varepsilon d\mu}{v_{\|}} B \;\; \mu B F.$$

The substitution of (31) into (39) leads to

$$W_f = \frac{1}{2}\int \frac{d^3x}{B}\left\{\frac{\partial B}{\partial \alpha}\left[\frac{\partial P}{\partial B}\frac{\partial B}{\partial \alpha} - \frac{\partial P}{\partial \alpha}\right]\left(\frac{\partial \eta}{\partial \theta}\right)^2 + \frac{\partial B}{\partial \theta}\left[\frac{\partial P}{\partial B}\frac{\partial B}{\partial \theta} - \frac{\partial P}{\partial \theta}\right]\left(\frac{\partial \eta}{\partial \alpha}\right)^2 - \right.$$

$$\left. - \frac{\partial B}{\partial \alpha}\left[\frac{\partial P}{\partial B}\frac{\partial B}{\partial \theta} - \frac{\partial P}{\partial \theta}\right]\frac{\partial \eta}{\partial \theta}\frac{\partial \eta}{\partial \alpha} - \frac{\partial B}{\partial \theta}\left[\frac{\partial P}{\partial B}\frac{\partial B}{\partial \alpha} - \frac{\partial P}{\partial \alpha}\right]\frac{\partial \eta}{\partial \theta}\frac{\partial \eta}{\partial \alpha}\right\}, \tag{40}$$

where $P = p_{\|} + p_\perp$,

$$P = \int \frac{d\varepsilon d\mu B}{v_{\|}} \;\; (M v_{\|}^2 + \mu B) F(\mu, \varepsilon, J).$$

The calculation of the derivatives of P, entering into (40), gives:

$$\frac{\partial P}{\partial B} = \int d\varepsilon d\mu F \;\; \frac{\partial}{\partial B}\left(B\frac{(M v_{\|}^2 + \mu B)}{v_{\|}}\right),$$

$$\frac{\partial P}{\partial \alpha} = \frac{\partial B}{\partial \alpha}\int d\varepsilon d\mu F \;\; \frac{\partial}{\partial B}\left(B\frac{(M v_{\|}^2 + \mu B)}{v_{\|}}\right) + \int \frac{d\varepsilon d\mu B}{v_{\|}} \;\; (M v_{\|}^2 + \mu B)\;\left(\frac{\partial F}{\partial \alpha}\right)_\varepsilon, \tag{41}$$

$$\frac{\partial P}{\partial \theta} = \frac{\partial B}{\partial \theta}\int d\varepsilon d\mu F \;\; \frac{\partial}{\partial B}\left(B\frac{(M v_{\|}^2 + \mu B)}{v_{\|}}\right) + \int \frac{d\varepsilon d\mu B}{v_{\|}} \;\; (M v_{\|}^2 + \mu B)\;\left(\frac{\partial F}{\partial \theta}\right)_\varepsilon.$$

Now reminding that $d^3x = d\alpha d\theta dl/B$, and accounting for (35), (36), (41), one can transform (40) to

$$W_f = \frac{1}{2}\int d\varepsilon d\mu d\alpha d\theta \left\{\frac{\partial J}{\partial \alpha}\left(\frac{\partial F}{\partial \alpha}\right)_\varepsilon\left(\frac{\partial \eta}{\partial \theta}\right)^2 + \right.$$

$$\left. + \frac{\partial J}{\partial \theta}\left(\frac{\partial F}{\partial \theta}\right)_\varepsilon\left(\frac{\partial \eta}{\partial \alpha}\right)^2 - \left(\frac{\partial J}{\partial \theta}\left(\frac{\partial F}{\partial \alpha}\right)_\varepsilon + \frac{\partial J}{\partial \alpha}\left(\frac{\partial F}{\partial \theta}\right)_\varepsilon\right)\frac{\partial \eta}{\partial \theta}\frac{\partial \eta}{\partial \alpha}\right\}. \tag{42}$$

The latter expression, together with equations

$$\left(\frac{\partial F}{\partial \theta}\right)_\varepsilon = \frac{\partial J}{\partial \theta}\left(\frac{\partial F}{\partial J}\right)_\varepsilon, \quad \left(\frac{\partial F}{\partial \alpha}\right)_\varepsilon = \frac{\partial J}{\partial \alpha}\left(\frac{\partial F}{\partial J}\right)_\varepsilon,$$

$$\left(\frac{\partial F}{\partial \theta}\right)_\varepsilon = \frac{\partial J}{\partial \theta}\left(\frac{\partial J}{\partial \alpha}\right)^{-1}\left(\frac{\partial F}{\partial \alpha}\right)_\varepsilon,$$

allows one to show easily that W_f equals to the expression (38) with the inverse sign, and hence the energy variation (23) is found to be identically zero.

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