15

ИНСТИТУТ ЯДЕРНОЙ ФИЗИКИ СО АН СССР

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НОВОСИБИРСК

Small Scale Chaos at Low Reynolds Numbers

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ABSTRACT

The small scale motion at low Reynolds number is shown to be excited through repeated nonlinear interactions and have chaotic phases. The latter property is used to derive the equations for the amplitudes. Solutions similar to those derived previously for turbulent fluctuations in the dissipation range are obtained. Properties of the short scale intermittency are analyzed. We show that no coherence and intermittency can be built up at the asymptotically high wave numbers.

1. INTRODUCTION

The energy of the fully developed turbulence is excited at some scale L, is transferred through the inertial range to vortices of the Kolmogorov scale η, where it is finally dissipated. Some portion of the turbulent energy penetrates into the dissipation range $k >> \eta^{-2}$ and produce a rapidly decreasing tail of the turbulence spectrum. In the dissipation range, the actual form of the turbulence spectrum is determined by vortices at low Reynolds number. Nevertheless, the pertinent nonlinearity of the small scale motions can not be discarded.

The linear stability analysis gives the following asymptote of the spectrum in the dissipation range (Townsend 1951, Novikov 1961)

$$F\ (k) \propto \exp\ [-\ (\eta k)^2]. \qquad (1.1)$$

In statistical theory it has been shown (Kraichnan 1959, Kuz'min 1971, Kuz'min & Patashinskii 1979, Dubovikov & Tatarskii 1986), that the energy transfer via a nonlinear cascade gives the more slowly decaying spectrum

$$F\ (k) \propto \exp\ (-\ \eta k). \qquad (1.2)$$

An attempt to solve the problem leads to difficulties, that are well known in theories with strong interactions. The main one is the failure of the perturbation theory. On the other hand, most of the additional difficulties, which are known to be peculiar to the inertial range turbulence theories, are absent at $\eta k \to \infty$. For example, the infrared divergences were not manifested, and the time proved to be a irrelevant variable in the dissipation range. Therefore the dissipation range turbulence is not as hard obstacle for the strong coupling techniques as one in the inertial range, and the renormalized perturbation expansions as well as renormalization group technique should be tested primarily in this area.

A number of effects in the system of dissipative modes have their own interest. In particular, in section 2, the self induced phase chaos of dissipative harmonics in flows of space dimensions $d \geq 2$ is considerred. This chaos is shown to be occur, and it prevents from producing of marked intermittency in the dissipation range. So the intermittency effects, which break the scaling in the inertial range, are of less and less importance at $\eta k \to \infty$. In this respect a considerable difference of the fluid turbulence from the one - dimensional systems should be noticed. In the latter problems Frisch & Morf 1981 revealed an enhanced influence of the intermittency at $\eta k \to \infty$. One may expect, that the scaling properties appear in a most pure fashion in fluid turbulence at $\eta k \to \infty$.

In section 3, a nonisotropic energy cascade to wave numbers $\eta k \gg 1$ is investigated, and a solution for the spectral tensor is obtained. In section 4, the expansion parameter of the renormalized diagram series is revealed. This parameter proved to be the energy conversion parameter, which is the nonlinear energy supply divided by the energy dissipation. The similar parameter in the inertial range (see Kuz'min & Patashinskii 1972) is equal to the squared Reynolds number determined from the effective viscosity. In the dissipation range, the Reynolds number is small, but the energy conversion parameter appear to be of the order of unity, because the nonlinear inflow of energy is equal approximately to the dissipation at a given scale. Thus the dissipation range turbulence is a typical example of a system with strong interaction. A reasonable theory can be obtained by taking into account only the first few diagrams. We believe, that other diagrams are of less importance in the renormalized series.

2. THE PHASE CHAOS AT SHORT SCALES

Let us consider the spatially periodic flow of incompressible fluid at small Reynolds number. The velocity field is represented as the Fourier series

$$\vec{u}\,(\vec{x},\ t) = \sum_{\vec{k}} \vec{u}\,(\vec{k},\ t)\ \exp\,(i\vec{k}\vec{x}),$$

$$\vec{u}\,(\vec{k},\ t) = L^{-d} \int d^d x\ \vec{u}\,(\vec{x},\ t)\ \exp\,(-\ i\vec{k}\vec{x}),$$

where L is the spatial period, which is supposed to be very large, and d is the dimension of space. From the Navier-Stokes equations, one obtains the equations for the complex amplitudes $\vec{u}(\vec{k}, t)$

$$(\partial/\partial t+\nu k^2)u_i(\vec{k},t)=(-i/2)P_{ijl}(\vec{k}) \sum_{\vec{k}} u_j(\vec{q}, t)u_l(\vec{k}-\vec{q}, t) \qquad (2.1)$$

where $P_{ijl}(\vec{k}) = k_j \Delta_{il}(\vec{k}) + k_l \Delta_{ij}(\vec{k})$, $\Delta_{ij}(\vec{k}) = \delta_{ij} - k_i k_j / k^2$,

$$\vec{k}\cdot\vec{u}(\vec{k}, t) = 0. \qquad (2.2)$$

We assume that the initial Fourier amplitudes $\vec{v}(\vec{k}) = \vec{u}(\vec{k}, t_0)$ differ from zero only when $k < k_0 = l^{-1}$, where $l \ll L$ is the main scale of the flow. Because the Reynolds number R is small, the subsequent evolution of the Fourier components at $k \le k_0$ is correctly determined by the equation (2.1) with the omitted right-hand side. The solution of the equation is

$$\vec{u}^0(\vec{k}, t) = \exp[-\nu k^2 (t - t_0)]\, \vec{v}(\vec{k}). \qquad (2.3)$$

At wave numbers $k > k_0$, the right-hand side of the equation (2.1) can not be discarded, because the nonlinear interactions serve as an energy source. In order to find nonlinear corrections to (2.3), we rewrite (2.1) as the integral equation

$$u_i(\vec{k}, t) = u_i^0(\vec{k}, t) + \int_{t_0}^{t} dt' \exp[-\nu k^2 (t - t')] \times$$

$$\times (i/2)\, P_{ijl}(\vec{k}) \sum_{\vec{q}} u_j(\vec{q}, t')\, u_l(\vec{k} - \vec{q}, t'). \qquad (2.4)$$

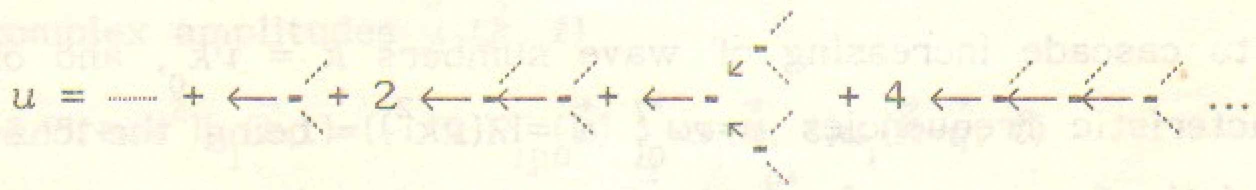
This equation can be simplified. The nonlinear interactions lead to cascade increasing of wave numbers $k = \nu k_0$, and of characteristic frequencies $\omega = \nu\omega_0$, $\omega_0 = 1/(\nu k^2)$ being the characteristic frequency of $\vec{u}^0$. At $k \gg k_0$, the time dependence of the velocities $u_j(\vec{q}, t')$, $u_l(\vec{k} - \vec{q}, t')$ in the right hand side of (2.4) is slow when compared to $\exp[-\nu k^2 (t - t')]$. Therefore, one can integrate over t' in (2.4) treating the velocities as time independent. This conjecture is supported by the detailed calculations performed by Kuz'min & Patashinskii, 1979. The simplified static equation for $\vec{u}(\vec{k})$ is then

$$u_i(\vec{k}) = v_i(\vec{k}) + \frac{1}{\nu k^2}\left(-\frac{i}{2}\right) P_{ijl}(\vec{k}) \sum_{\vec{q}} u_j(\vec{q})\, u_l(\vec{k}-\vec{q}). \qquad (2.5)$$

We use the graphic notations similar to those used by Wyld 1961, Kuz'min & Patashinskii, 1979. The function $(\nu k^2)^{-1}$ is represented by an arrow $\longleftarrow$. The vertex operator $(-i/2)P_{ijl}(\vec{k}) \sum_{\vec{q}}$ is represented by a point $-$, and the large scale component $\vec{v}$ is represented by a line $\cdots\cdots$. Thus the equation (2.5) can be written symbolically as

$$\vec{u} = \cdots\cdots + \longleftarrow - \genfrac{}{}{0pt}{}{\vec{u}}{\vec{u}}\,. \qquad (2.6)$$

Iterating (2.5), (2.6), one obtains the velocity $\vec{u}(\vec{k})$ as a series in its large scale component $\vec{v}(\vec{k})$. The effective small parameter of the expansion is the Reynolds number R. The graphical form of the series is given by a sum of tree diagrams


... (2.7)

At each vertex the wave vector is conserved. The sum of the wave vectors of entering lines is equal to the wave vector of exiting arrow, so the wave vector flows without any loss from the brunches —— = $\vec{v}$ to the trunk of a tree.

The wave vector of a diagram of n^{th} order is equal to the sum of the wave vectors of all factors $\vec{v}\,(\vec{k}_i)$, where $k_i \propto k_0$

$$\vec{k} = \sum_{i=1}^{n} \vec{k}_i, \qquad k_i \propto k_0. \tag{2.8}$$

The analytic expression for the n^{th} order diagram is of the form

$$I_n = \sum_{\vec{k}_1+\ldots+\vec{k}_n=\vec{k}} M(\vec{k}_1, \vec{k}_2, \ldots, \vec{k}_n)\Pi(\vec{k}_1, \vec{k}_2, \ldots, \vec{k}_n), \tag{2.9}$$

where $\Pi = \prod_{i=1}^{n} \vec{v}\,(\vec{k}_i)$, M is a vertex function of n^{th} order, which is composed of the functions $(\nu k^2)^{-1}$, and of the vertex functions $P(\vec{k})$. Indices are not shown for simplicity.

Let us treat each $\vec{k}_i$ in (2.8) as a step, and the sum (2.8) as a result of a walk in the Fourier space. The sum (2.9) over all $\vec{k}_i$ is thus a sum of contributions to $\vec{u}\,(\vec{k})$

from different paths $\{\vec{k}_i\}$. For $k \gg k_0$, the first $n < k/k_0$ terms in the expansion (2.7) give no contribution to $\vec{u}\,(\vec{k})$, because the condition (2.8) can be fulfilled only if $n > k/k_0$.

Contribution of terms, which order n exceeds k/k_0 only slightly, is still small, because the available volume, which is restricted by (2.8), is small. On the other hand, the contribution of terms of orders $n \gg k/k_0$ is small in the parameter $R \ll 1$. So there exist an optimal order $n_0 > k/k_0$ giving the maximal contribution to $\vec{u}\,(\vec{k})$. The optimal order n_0 is produced by a competition among the available volume and the power of the effective expansion parameter.

It may be concluded, that the optimal n_0 corresponds to such paths, that almost every step leads in $\vec{k}$-direction, so that the longitudinal projections of $\vec{k}_i$ are positive, and are of order of k_0. The transversal components of $\vec{k}_i$ are of the same order but have no preferred direction.

When estimating the expression (2.9), the phases of the complex amplitudes have to be taken into account. Denoting $\vec{v}_m(\vec{k}) = |\vec{v}_m(\vec{k})| \exp[i\phi_m(\vec{k})]$, one has $\Pi = |\Pi| \exp(i\Phi)$, where $\Pi = \exp\left[\sum_{i=1}^{n} \log|\vec{v}(\vec{k}_i)|\right]$, $\Phi = \sum_{i=1}^{n} \phi(\vec{k}_i)$. Let us suppose, that $\vec{v}(\vec{k})$ is an analytic function of $\vec{k}$. For a small variation of a path $\{\vec{k}_i + \delta\vec{k}_i\}$, the phase Φ changes additively

$$\delta\Phi = \sum_{i=1}^{n} \delta\phi(\vec{k}_i), \qquad \delta\phi(\vec{k}_i) = \left.\frac{\partial\phi(\vec{k})}{\partial\vec{k}}\right|_{\vec{k}=\vec{k}_i} \delta\vec{k}_i. \tag{2.10}$$

Thus at large n, a small variation of a path may lead to a great variation of the total phase $\delta\Phi > \pi$. Such a behavior of Φ implies a strong interference of contributions from different paths. Only variations inside a thin tube in $\vec{k}$-space are allowable without destroying the phase Φ.

Let us estimate the effective number of tubes with different phases. The total shift of the phase (2.10) is composed of a large number of small shifts $\delta\phi(\vec{k}_i) \propto \delta\vec{k}_i/k_0$ with arbitrary signs, so $\delta\Phi$ is estimated as in the theory of Brownian motion as $\delta\Phi \propto (\delta\vec{k}_i/k_0)\sqrt{n}$. This value is less than π if $\delta k_i < k_0/\sqrt{n}$. Thus in (2.9) one may replace the sum over k_i, $i=1,2,\dots$ by a sum over elementary cubes of volume $(\delta k_i)^d \propto (k_0/\sqrt{n})^d \propto (k_0^3/k)^{d/2}$ (note, that $n \propto k/k_0$). The volume in which the factors $v(\vec{k})$ do not vanish is of order k_0^d, so the number of such cubes is equal to $k_0^d/(k_0^3/k)^{d/2} \propto$ $\propto (k/k_0)^{d/2}$.

Any two paths are considered as different only if they pass through different sets of cubes in any sequence, so any path occur in (2.9) $n! \approx 2\pi n^{n+1/2}/\exp(n)$ times. Thus, the number of different paths $N(k)$ is of the order of

$$N(k) \propto [(k/k_0)^{d/2}]^n/n! \propto (k/k_0)^{n(d-2)/2} \exp(n). \qquad (2.11)$$

At $d \geq 2$, $n \propto k/k_0 >> 1$ this number is very large.

The expression (2.9) can be written as a sum of contributions from the tubes $\{\vec{k}_i\}$

$$I_n\{\vec{k}\} = \sum_{\{\vec{k}\}} M\{\vec{k}_i\}\ \Pi\{\vec{k}_i\}. \qquad (2.12)$$

The phase of the contributions has been shown to be a sharp and complicated function of the path $\{\vec{k}\}$. Very often such a complicated function with sharp and unpredictable behavior is identified to a random function (see for example Lichtenberg & Lieberman 1983). Summing up (2.12) of such random contributions may be treated as a random walk in a complex plane. Both the amplitude and the phase of I_n, which are results of the random walk, are random.

From the above considerations, we suppose that only the statistical properties of the complex Fourier amplitudes $\vec{u}(\vec{k})$ do matter at large wave numbers. The memory about the phases of $\vec{v}(\vec{k})$ is lost, when the energy transfers to wave numbers $k >> l^{-1}$. The same supposition seems reasonable for the most of the characteristics of the amplitudes of $\vec{v}(\vec{k})$. However, some information about the orientation of the initial vortex is conserved because the random walk in $\vec{k}$-space has the preferred direction $\vec{k}$.

If the most of the information about the large scale field $\vec{v}$ is lost, we may replace it by a random field with suitable statistical characteristics. For the dissipation range turbulent fluctuations, such a theory was studied previously by Kraichnan 1959, Kuz'min 1971, Kuz'min & Patashinskii 1979, Dubovikov & Tatarskii 1986 with the result (1.2). We develop much the similar universal theory for the small scale motions in vortices at small Reynolds number. The only complication is the loss of isotropy.

3. THE EXPONENTIAL SOLUTION TO THE EQUATION FOR THE SPECTRAL TENSOR

In the previous section, we examined a short wave asymptote of the Fourier transformed velocity at small Reynolds number. We argued that all details but isotropy of the large scale velocity do not affect the small scale component at $d \geq 2$. So the initial dynamic problem may be replaced by a more simple statistical one.

Let us consider the equation (2.5), where $\vec{v}(\vec{k})$ is now an external random field, the source of the small scale motion. It is assumed, that the random field $\vec{v}$ is homogeneous, has normal distribution, but is not isotropic. Note, that this assumption is not valid in the theory of developed turbulence at asymptotically high Reynolds numbers, because of the intermittency at the Kolmogorov scale (Monin & Yaglom 1971). In particular, the local Kolmogorov scale η may fluctuate, and averaging over the fluctuations generally influence the spectrum (Kraichnan 1967, Keller & Yaglom 1970). On the other hand, our consideration is restricted by the condition $R \ll 1$. Our choice of the ensemble is rather related to the structure of an individual vortex packet that has the same $<|\vec{v}(\vec{k})|^2>$.

For the Gaussian field $\vec{v}(\vec{k})$ any mean value of the type

$$<v_{i_1}(\vec{k}_1) \dots v_{i_n}(\vec{k}_n)>$$

can be represented by a sum of products of all possible pairwise averages (the analog of the Wick theorem in the quantum field theory). The average $<v_i(\vec{k})\ v_j(\vec{k}')>$ is represented by the Hermitian spectral tensor

$$F^0_{ij}(\vec{k}) = (L/2\pi)^d <v_i(\vec{k})\ v_j(-\vec{k})>.$$

We assume that the spectral tensor differs from zero only when $k \leq l^{-1}$.

Any velocity function can be expanded in a formal functional series in $\vec{v}(\vec{k})$. The example of such an expansion is the diagram series (2.7). The diagram expansion for the spectral tensor

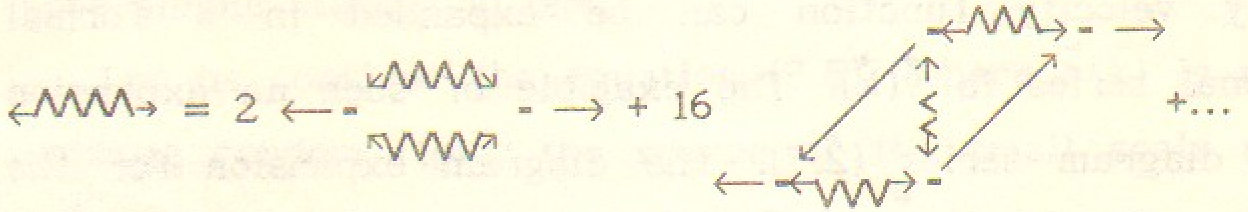
$$F_{ij}(\vec{k}) = (L/2\pi)^d <u_i(\vec{k}) u_j(-\vec{k})>$$

is obtained after multiplying (2.7) by the similar expansion for $u_j(-\vec{k})$ and averaging over $\vec{v}(\vec{k})$. In the limit $L \to \infty$, any sums over wavevectors are replaced by integrals according to

$$(2\pi/L)^d \sum_{\vec{q}} \Rightarrow \int d^d q.$$

After partial summing up of the nonrenormalized diagram series, one arrives at the complete system of diagram equations for the spectral tensor F_{ij}, the response tensor and for the vertex functions (Kuz'min & Patashinskii 1979).

The analysis of the equations is similar to that performed by Kuz'min & Patashinskii 1979 for isotropic dissipation range. The response tensor and the vertices describe the external nonrandom perturbations which are unaffected by the weak small scale component $\vec{u}(\vec{k})$, so these functions are assumed to coincide with the nonrenormalized ones. In other words, for $k \gg l^{-1}$ the nonlinearity should be taken into account only so far as it is the only energy source. Therefore, it remains to solve the equation for the spectral tensor F_{ij}. This equation assumes the form (see eq. (8) of Kuz'min & Patashinskii 1979)


(3.1)

where the spectral tensor is represented by a wavy line. We shall seek the solution to the equation (3.1) in the form

$$F_{ij}(\vec{k})=\Psi_{ij}(\vec{k}) \exp\{-[\eta(\vec{e})k]^{\gamma}\}. \tag{3.2}$$

Here $\vec{e} = \vec{k}/k$ is the unit vector in the direction of $\vec{k}$. The function $\eta(\vec{e})$ is supposed to be determined by the condition $F_{ij} \approx F^{0}_{ij}$ at $k \approx l^{-1}$. We suppose, that $\eta(\vec{e})=\eta(-\vec{e})$, $\gamma>1$, Ψ_{ij} is a Hermitian tensor that varies, when $k \gg l^{-1}$, not more rapidly than a power function.

Let us compute approximately the tensor F_{ij} with the aid of Eq. (3.1) on the right-hand side of which we retain only the first diagram. The equation to be solved is

$$F_{ij}(\vec{k}) = \nu^{-2}k^{-4}\int d^{d}q[k_{l}F_{lm}(\vec{q})k_{m}\ \Delta_{is}(\vec{k})F_{sn}(\vec{k}-\vec{q})\Delta_{nj}(\vec{k}) + $$
$$+k_{l}F_{lm}(\vec{q})\Delta_{mj}(\vec{k})\ \Delta_{is}(\vec{k})F_{sn}(\vec{k}-\vec{q})k_{n}].$$

Substituting (3.2) into this equation, we have

$$\Psi_{ij}(\vec{k})=\nu^{-2}k^{-4}\int d^{d}q\ [k_{l}\Psi_{lm}(\vec{q})k_{m}\ \Delta_{is}(\vec{k})\Psi_{sn}(\vec{k}-\vec{q})\Delta_{nj}(\vec{k})+$$
$$+k_{l}\Psi_{lm}(\vec{q})\Delta_{mj}(\vec{k})\ \Delta_{is}(\vec{k})\Psi_{sn}(\vec{k}-\vec{q})k_{n}]\ \exp(K), \tag{3.3}$$

where $K = [k\eta(\vec{k})]^{\gamma} - [q\eta(\vec{q})]^{\gamma} - [|\vec{k}-\vec{q}|\eta(\vec{k}-\vec{q})]^{\gamma}$. At $\eta k \gg 1$, the dominant contribution to the integral is made by the region, where the index of the exponential function has its maximum value. For $\gamma > 1$, $\eta(\vec{e}) = \eta =$ const, the maximum lies in the region, where $\vec{q} = \vec{k}-\vec{q} = \vec{k}/2$. It is clear, that this maximum remains if nonisotropy is not too large. To define this condition more precisely, let us expand the index of the exponential function in the components of the wave vectors, which is transverse to $\vec{k}$. If we denote the nondimensional longitudinal and transversal component of $\vec{q}$ as $s = \vec{e}(\vec{e}\cdot\vec{q}/k)$, and $w_{l} = \Delta_{lj}(\vec{e})\ q_{j}/k$, then

$$\eta^{\gamma}(\vec{q}/q) \approx \eta^{\gamma}(\vec{e}) + \frac{\partial\eta^{\gamma}}{\partial e_{m}} w_{m}/s + \frac{1}{2}\frac{\partial^{2}\eta^{\gamma}}{\partial e_{m}\partial e_{n}} w_{m}w_{n}/s^{2}, \tag{3.4}$$

$$\eta^{\gamma}[(\vec{k}-\vec{q})/|\vec{k}-\vec{q}|] \approx \eta^{\gamma}(\vec{e}) - \frac{\partial\eta^{\gamma}}{\partial e_{m}} w_{m}/(1-s) + \frac{1}{2}\frac{\partial^{2}\eta^{\gamma}}{\partial e_{m}\partial e_{n}} w_{m}w_{n}/(1-s)^{2},$$

$$q^{\gamma} \approx s^{\gamma}k^{\gamma}[1+\gamma w^{2}/(2s^{2})],\quad |\vec{k}-\vec{q}|^{\gamma} \approx (1-s)^{\gamma}k^{\gamma}\{1+\gamma w^{2}/[2(1-s)^{2}]\},$$

and

$$K = k^{\gamma}\eta^{\gamma}(\vec{e})\Big\{[1-s^{\gamma}-(1-s)^{\gamma}]+[(1-s)^{\gamma-1}-s^{\gamma-1}]\,\frac{1}{\eta^{\gamma}}\frac{\partial\eta^{\gamma}}{\partial e_m}\,w_m - \tag{3.5}$$

$$- \frac{1}{2}\,[s^{\gamma-2} + (1-s)^{\gamma-2}]\Big\}\left(\frac{1}{\eta^{\gamma}}\frac{\partial^2\eta^{\gamma}}{\partial e_m\,\partial e_n} + \gamma\delta_{mn}\right) w_m w_n .$$

It is convenient to choose the special coordinate system in which the last axis is directed along $\vec{k}$ and the other axes are directed along the eigenvectors of the matrix

$$A_{mn} = \gamma\delta_{mn} + \frac{1}{\eta^{\gamma}}\frac{\partial\,\eta^{\gamma}}{\partial e_m\,\partial e_n} . \tag{3.6}$$

In this coordinate system the matrix A looks like

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 & \ldots\ldots & 0 \\ & \lambda_2 & 0 & \ldots\ldots & 0 \\ \ldots & \ldots & \ldots & \ldots\ldots & \ldots \\ 0 & 0 & 0 & \ldots\lambda_{d-1} & 0 \\ 0 & 0 & 0 & \ldots 0 & \gamma \end{pmatrix}$$

where $\lambda_1,\lambda_2,\ldots,\lambda_{d-1}$, γ are the eigenvalues of the matrix A.

One sees, that if the eigenvalues $\lambda_1,\ldots,\lambda_{d-1}$ have different signs, then the function K has a saddle point at $\vec{w}=0$, $s=1/2$. If all λ_m are positive, the function K has a maximum at this point. If any $\lambda_m=0$, the terms of higher order in expansions (3.4) should be taken into account.

From (3.6) it follows, that an eigenvalue λ_m may be negative if the second derivative of η^{γ} in the associated direction is negative and is sufficiently large in its absolute value, that is

$$\frac{1}{\eta^{\gamma}}\frac{\partial^2\eta^{\gamma}}{\partial e_m^{\,2}} < -\,\gamma .$$

In these cases the dominant contribution to the integral (3.3) comes from the Fourier harmonics with strongly noncollinear wave vectors. Therefore, a strong interactions among the Fourier modes with different directions of their wave vectors occur. These interactions tend to diminish the strong initial nonisotropy while the energy is transferred to high wave number region. One may suppose, that the strong nonisotropy with negative eigenvalues does not occur at $\eta k \to \infty$, though it might take place at a moderate ηk. On the contrary, arbitrary nonisotropy with positive eigenvalues is possible at any $\eta k \gg 1$. We shall not consider these cases in more details.

For a moderate degree of nonisotropy, the eigenvalues are positive and the dominant contribution to the integral (3.3) comes from the region where $\vec{q}\approx\vec{k}/2$. Thus, the right-hand side of Eq. (3.3) is of order $exp\ [(\eta k)^{\gamma}(1 - 2^{1-\gamma})]$, and is exponentially large as compared to the left-hand side. So for $\gamma>1$ the equation (3.3) can not be satisfied. For $0<\gamma<1$, the index of the exponential function has a maximum, when $\vec{q} \ll \vec{k}$ or $|\vec{k}-\vec{q}| \ll k$. This corresponds to a case in which the dominant role is played by interactions of the short-wave pulsations directly with the pulsations of the principal scale. It was however, been shown (Townsend 1951, Novikov 1961), that such interactions lead to a solution with $\gamma=2$, and not with $\gamma<1$.

Therefore, the only γ value that is not at variance with the equation is $\gamma=1$.

For $\gamma=1$, the expressions (3.5), (3.6) look like

$$K = -\frac{1}{2}\frac{\eta k}{s(1-s)} A_{mn} w_m w_n , \quad (3.7)$$

$$A_{mn} = \delta_{mn} + \frac{1}{\eta}\frac{\partial^2 \eta}{\partial e_m \partial e_n} . \quad (3.8)$$

The preexponential factor Ψ is obtained with the aid of the Laplace method (Erdelyi 1961). The exponent K contains the large factor $\eta k \gg 1$, and the dominant contribution to the integral in the right-hand side of Eq.(3.3) is made by the region, where $\vec{q}$ and $\vec{k}$ are almost collinear. So one may expand $\Psi_{lm}(\vec{q})$, $\Psi_{sn}(\vec{k}-\vec{q})$ in (3.3) in powers of $\vec{w}$

$$\Psi_{lm}(\vec{q}) = \Psi_{lm}(s\vec{k}) + k \left.\frac{\partial \Psi_{lm}(\vec{q})}{\partial q_s}\right|_{w=0} w_s + \frac{1}{2} k^2 \left.\frac{\partial^2 \Psi_{lm}(\vec{q})}{\partial q_s \partial q_r}\right|_{w=0} w_s w_r + \ldots \quad (3.9)$$

$$\Psi_{lm}(\vec{p}) = \Psi_{lm}[(1-s)\vec{k}] - k \left.\frac{\partial \Psi_{lm}(\vec{p})}{\partial p_s}\right|_{\vec{w}=0} w_s + \frac{1}{2} s^2 \left.\frac{\partial^2 \Psi_{lm}(\vec{p})}{\partial p_s \partial p_r}\right|_{\vec{w}=0} w_s w_r - \ldots$$

where

$$\vec{q} = k(s\vec{e} + \vec{w}), \quad \vec{p} = k[(1-s)\vec{e} - \vec{w}], \quad \vec{e} = \vec{k}/k. \quad (3.10)$$

The solenoidality condition (2.2) implies that

$$\Psi_{ij}(\vec{q}) q_j = \Psi_{ij}(\vec{p}) p_j = 0. \quad (3.11)$$

Substituting (3.9), (3.10) into (3.11) and equating the terms with equal powers of w, one finds, that

$$k_m \left.\frac{\partial \Psi_{im}(\vec{q})}{\partial q_s}\right|_{\vec{w}=0} w_s = -\frac{1}{s}\Psi_{im}(s\vec{k}) w_m ,$$

$$\frac{1}{2} k_m \left.\frac{\partial^2 \Psi_{im}(\vec{q})}{\partial q_s \partial q_r}\right|_{\vec{w}=0} w_s w_r = -\frac{1}{s} \left.\frac{\partial \Psi_{im}(\vec{q})}{\partial q_s}\right|_{\vec{w}=0} w_s w_m , \quad (3.12)$$

$$k_m \left.\frac{\partial \Psi_{im}(\vec{p})}{\partial p_s}\right|_{\vec{w}=0} w_s = -\frac{1}{1-s}\Psi_{im}[(1-s)\vec{k}] w_m .$$

By inserting (3.7)-(3.12) into (3.3), one obtains

$$\Psi_{ij}(\vec{k}) = \frac{1}{\nu^2 k^2}\int d^d q \left\{\frac{1}{s^2}\Psi_{mn}(s\vec{k})\Psi_{ij}[(1-s)\vec{k}] - \frac{1}{s(1-s)}\Psi_{mj}(s\vec{k})\Psi_{in}[(1-s)\vec{k}]\right\} w_m w_n \exp\left[-\frac{1}{2}\frac{\eta k}{s(1-s)} A_{rt} w_r w_t\right]. \quad (3.13)$$

The integration over $\vec{w}$ gives

$$\int d^{d-1} w \, w_m w_n \exp\left[-\frac{1}{2}\frac{\eta k}{s(1-s)} A_{rt} w_r w_t\right] = \quad (3.14)$$

$$= \delta_{mn} \frac{\pi^{(d-1)/2}[2s(1-s)]^{(d+1)/2}}{2\lambda_n \sqrt{\lambda_1 \lambda_2 \ldots \lambda_{d-1}}\,(\eta k)^{(d+1)/2}} .$$

By virtue of (3.14), Eq.(3.13) assumes the form

$$\Psi_{ij}(k) = \frac{(2\pi)^{(d-1)/2} k^{d-2}}{\nu^2 \sqrt{\lambda_1 \ldots \lambda_{d-1}}\,(\eta k)^{(d+1)/2}} \int ds [s(1-s)]^{(d+1)/2} \times$$

$$\times \sum_{m=1}^{d-1}\left\{\frac{1}{s^2\lambda_m}\Psi_{mm}(s\vec{k})\Psi_{ij}[(1-s)\vec{k}] - \frac{1}{s(1-s)\lambda_m}\Psi_{mj}(s\vec{k})\Psi_{im}[(1-s)\vec{k}]\right\}.$$

This equation has the power solution

$$\Psi_{ij}(\vec{k}) = \frac{30\nu^2(\eta k)^{(d+1)/2}\sqrt{\lambda_1\lambda_2\ldots\lambda_{d-1}}}{2^{(d-3)/2}(3d-5)k^{d-2}\,\pi^{(d-1)/2}}\begin{pmatrix} \lambda_1 & 0 & 0 & \ldots & 0\\ 0 & \lambda_2 & 0 & \ldots & 0\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ 0 & 0 & 0 & \ldots\lambda_{d-1} & 0\\ 0 & 0 & 0 & \ldots 0 & 0 \end{pmatrix}$$

Therefore, in any coordinate system the spectral tensor (3.2) can be written as

$$F_{ij}(\vec{k})=\exp[-\eta(\vec{k}/k)]\frac{30\nu^2(\eta k)^{(d+1)/2}\sqrt{\mathrm{Det}A}}{2^{(d-3)/2}(3d-5)k^{d-2}\pi^{(d-1)/2}}\Delta_{im}(\vec{k})A_{mn}\Delta_{nj}(\vec{k}) \qquad (3.15)$$

where the matrix $A_{mn}=A_{mn}(\vec{k}/k)$ is defined by the Eq.(3.8). If $\eta(\vec{k}/k)$ does not depend on the direction of $\vec{k}$, then $A_{mn}=\delta_{mn}$, and (3.15) reduces to the solution obtained by Kuz'min & Patashinskii 1979, Kuz'min 1979 in the same approximation.

4. EXPANSION PARAMETER AND INTERMITTENCY FACTOR IN THE DISSIPATION RANGE

Let us consider diagrams of more high order. A diagram F_n, containing n integrations over d^dk, has n+1 wavy lines, $2n$ vertices and $2n$ functions $(\nu k^2)^{-1}$.

$$F_n \propto (P/\nu k^2)^{2n}\, F^{n+1}[k/(n+1)]\, (k_\perp^{d-1}k)^n,$$

where $k_\perp$ is the size of the integration domain in the transverse plane. The exponential factor restricts this size

and $k_\perp$ can be estimated as

$$k_\perp \propto \sqrt{kk_0}.$$

One sees, that the effective parameter of expansion (3.1) has the same order of magnitude as the first diagram in the right-hand side divided by the spectral function F. This parameter can be written as

$$\mu \propto \frac{F^2(k/2)[P^2/(\nu k^2)]k_\perp^{d-1}k}{F(k)\,(\nu k^2)}.$$

The numerator is the nonlinear supply of energy, and the denominator is the viscous dissipation. In a quasi-steady case, these factors have the same value, so $\mu \propto 1$.

Kuz'min & Patashinskii 1972, 1978 revealed a similar parameter in the inertial range. The turbulent medium was regarded as been made of wave packets. The Kolmogorov scaling was treated as a situation wherein the wave packets of all scales are constructed in a similar fashion and lose an equal amount of power when overcoming the turbulent viscosity. So in inertial range both the numerator and the denominator in μ are constants separately. The factor μ should be naturally called the energy conversion parameter. The parameter μ proved to be of the order of the Reynolds number of the wave packets, determined from the effective viscosity. The Kolmogorov scaling corresponds to a case wherein this Reynolds number does not depend on scale and is an universal constant.

In the dissipation range the Reynolds number of the wave packets is very small, but the energy conversion parameter μ does not. The actual role of high order diagrams is estimated by a direct calculation, which is possible in the dissipation range theory. The readers may be referred to our analysis of isotropic dissipation range (Kuz'min & Patashinskii 1979), where it was shown, that the approximate solution changes little when the next term in the series (3.1) is taken into account.

Similarly to Kuz'min 1979, let us consider the small scale intermittency in the framework of the diagram technique. The small scale component of the velocity field in the usual space is defined as

$$\vec{u}(\vec{x},\ell) = \sum_{\Omega(\ell)} d^d k\ \vec{u}(\vec{k}) \exp(i\vec{k}\vec{x}).$$

The sum is over the region $\Omega(\ell)$ where $k > \ell^{-1}$. The intermittency of the small scale velocity $\vec{u}$ $(\vec{k}, \ell)$ is determined by the flatness factor (Monin & Yaglom 1971)

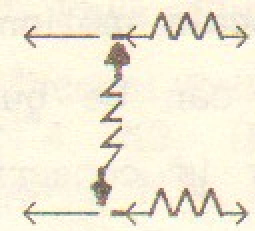
$$\kappa(\ell) = [\langle u_i(\vec{x}, \ell)^4\rangle - 3\langle u_i(\vec{x},\ell)^2\rangle^2]/\langle u_i(\vec{x}, \ell)^2\rangle^2. \qquad (4.1)$$

Let us consider the Fourier expansion of the numerator $a(\ell)$ in (4.1)

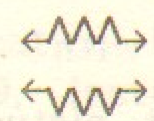
$$a(\ell)=\sum_{\vec{k}_1} \ldots \sum_{\vec{k}_4} \exp[i(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)\vec{x}][\langle u_i(\vec{k}_1)u_i(\vec{k}_2)u_i(\vec{k}_3)u_i(\vec{k}_4)\rangle -$$

$$-3\ \langle u_i(\vec{k}_1)u_i(\vec{k}_2)\rangle\ \langle u_i(\vec{k}_3)u_i(\vec{k}_4)\rangle],$$

where the summing is performed over all $\vec{k}_1 \ldots \vec{k}_4$ in $\Omega(\ell)$. After substituting of (2.7) into (4.2), one obtains $a(\ell)$ as a series of all possible diagrams with four exiting lines. One of the lowest order diagrams is



The spectral tensor decreases rapidly as its wave number increases. Therefore the dominant contribution to the sum (4.2) comes from the region where the wave number of the internal wavy line is of order of η^{-1}. When compared to the denominator $b(\ell)$ in μ

$b(\ell) \propto$ 

$a(\ell)$ contains an additional wavy line, two bare Green functions $\longleftarrow \propto (\nu k^2)^{-1}$, two vertex operators $\propto k$ and one summing over the wave vectors $\vec{k} \propto \eta^{-1}$. Therefore the flatness is of the order

$$\kappa(\ell) = a(\ell)/b(\ell) \propto (\ell/\eta)^2 \ll 1.$$

The similar conclusion follows from the analysis of the diagrams of more high orders. So the diagram technique reproduces the above conclusion concerning the intermittency in the dissipation range, and the solution for the spectral tensor is self-consistent.

5. CONCLUSION

In this paper we have analyzed the energy cascade process at asymptotically high wave numbers. We give general arguments, that the small scale motions should have random phases, so no intermittency can be built up in this region. Some amount of nonisotropy is conserved, while the energy cascades to high wave numbers. It may be supposed, that the motion in the dissipation range is composed of two components. The first one is produced by decaying coherent vortices of Kolmogorov scale. The energy spectrum of this component may be similar to (1.1), which may be modified by intermittency effects. The second one is the universal incoherent component with the spectrum (1.2). Some similarity of the present picture to that proposed by Benzi *et al* 1986 from direct computer simulations of the two-dimensional flows should be noted. The universal analytical theory for the incoherent component is proposed and the solution for the spectral tensor is obtained.

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