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INVERSE SPECTRAL TRANSFORM FOR THE MODIFIED KADOMTSEV—PETVIASHVILI EQUATION

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A B S T R A C T

The 2+1-dimensional modified Kadomtsev — Petviashvili (mKP) equation is studied by the inverse spectral transform method. The initial-value problems for the mKP-I and mKP-II equations are solved by the nonlocal Riemann — Hilbert and $\bar{\partial}$-problems technique for the initial data decaying sufficiently rapidly at infinity. The lump solutions for the mKP-I equation are found explicitly. Wide classes of the exact solutions for the mKP equation, namely, the rational solutions, including the plane lumps for the mKP-I equation, solutions with functional parameters, the plane solitons and breathers are constructed by the use of the $\bar{\partial}$-dressing method based on the nonlocal $\bar{\partial}$-problem. The Miura transformation between the mKP and KP equations is discussed.

Contents

1. INTRODUCTION

Wide classes of partial differential equations have been studied in detail by the inverse spectral (scattering) transform (IST) method (see, eq. [1—6]). The Korteweg de Vries (KdV) equation, the modified Korteweg de Vries (mKdV) equation, the nonlinear Schrödinger (NLS) equation and the Heisenberg ferromagnet model equation are one of the most important and interesting representatives of the 1+1-dimensional IST integrable equations both from the physical and mathematical points of view [1—6].

All these four equations have the 2+1-dimensional (two spatial and one temporal coordinates) integrable generalizations. They are the Kadomtsev—Petviashvili (KP) equation, the modified Kadomtsev—Petviashvili (mKP) equation, the Davey—Stewartson (DS) equation and the Ishimori equation respectively. Actual integration of the KP equation has been connected with the essential generalization of the IST method, namely, with the introducing the nonlocal Riemann—Hilbert problem and $\bar{\partial}$-problem into the method [7—9]. Then the DS equation [10—13], the Ishimori equation [14—16] and some other 2+1-dimensional equations have been solved. The nonlocal Riemann—Hilbert and $\bar{\partial}$-problems method now is basic tool for solving the 2+1-dimensional integrable equations (see e. q. the reviews [17—19]). In parallel the very general $\bar{\partial}$-dressing method has been proposed [20—22]. With the use of all these methods the KP, DS, and Ishimori equations have been analyzed in detail.

The aim of the present paper is to study the last (or the second) equation from the quartet of the $2+1$-dimensional equations mentioned above, namely, the mKP equation. This equation is of the form

$$V_t+V_{xxx}-\frac{3}{2}V^2V_x+3\sigma^2\partial_x^{-1}V_{yy}-3\sigma V_x\partial_x^{-1}V_y=0, \quad (1.1)$$

where $\sigma^2=\pm 1$. Equation (1.1) has been introduced in [23] within the framework of the gauge—invariant description of the KP equation. In [24] it has appeared as the first member of the 1-st modified KP hierarchy. Equation (1.1) is equivalent to the compatibility condition for the linear system [25].

$$\sigma\Psi_y+\Psi_{xx}+V\Psi_x=0, \quad (1.2a)$$

$$\Psi_t+4\Psi_{xxx}+6V\Psi_{xx}+\left(3V_x-3\sigma\partial_x^{-1}V_y+\frac{3}{2}V^2\right)\Psi_x+\alpha\Psi=0, \quad (1.2b)$$

where α is an arbitrary constant.

Generally, solutions of equation (1.1) are the complex-valued functions. But in the case $\sigma^2=-1$ $(\sigma=i)$ it admits the reduction to the pure imaginary V while at $\sigma^2=1$ $(\sigma=1)$ there is an obvious reduction to real V. In view of this it is natural to introduce a new dependent variable u defined by $V=\sigma u$. In terms of u equation (1.1) looks like

$$u_t+u_{xxx}-3\sigma^2\left(\frac{1}{2}u^2u_x-\partial_x^{-1}u_{yy}+u_x\partial_x^{-1}u_y\right)=0. \quad (1.3)$$

In what follows we will refer to equation (1.3) as the mKP equation, namely, as the mKP-I equation at $\sigma=i$ $(\sigma^2=-1)$ and as the mKP-II equation at $\sigma=1$. Both the mKP-I and the mKP-II equations obviously admit the reduction to real u.

The mKP equation is of the great interest by several reasons. Firstly this equation may be relevant for the description of the water waves on the plane (x, y) in a situation when similar to the mKdV case one should take into account the cubic nonlinearity. Secondly there is a close algebraic interrelation between the mKP and KP equations similar the KdV case. In particular, they are related by the two-dimensional Miura transformation [23, 25]

$$u_{KP}=-\frac{1}{2}\sigma\partial_x^{-1}V_y-\frac{1}{2}V_x-\frac{1}{4}V^2.$$

Thirdly, the mKP equation, more exactly, the problem of finding the exact solutions of the linear equation (1.2a) arises within the problem of construction of the exponentially localized solitons for the Ishimori equation [26]. At last, the case of the mKP equation is a very interesting one from the point of view of the IST method itself. Indeed, the linear equation (1.2a) is not an appropriate reduction of the good 2×2 matrix linear problem in contrast to the mKdV case. So, one should apply the IST technique directly to the scalar linear problem (1.2a) with the nonstandard normalization, i.e. with the nontrivial coefficient in front of the first order derivative. Thus the study of the mKP equation seems to be of an essential importance for the development of the IST method.

In this paper we will consider both the initial-value problem and the problem of constructing of classes of exact solutions for the mKP equations (1.3). The general approach is the similar to that for the KP equation [8, 9, 17—19]. But the mKP equation has several important features. The first and main one is that in the mKP case the adequate introduction of the spectral parameter λ is achieved by the transition to the function χ via

$$\Psi(x, y, t; \lambda)=\chi(x, y, t; \lambda)\exp\left(i\frac{x}{\lambda}+\frac{y}{\sigma\lambda^2}\right). \quad (1.4)$$

The function χ defined by (1.4) admits the canonical normalization $\chi \xrightarrow[\lambda\to\infty]{} 1$ and what is more important it obeys equations (1.2) for any constraint free inverse problem data. Other features of the mKP equation are discussed in the text.

We will solve the initial-value problem for the mKP equation for the initial data $u(x, y, 0)$ decreasing at $(x^2+y^2)^{1/2}\to\infty$ and obeying the condition $\int_{-\infty}^{+\infty} dx\, u(x, y, 0)=0$. Similar to the KP case the inverse problem equations for the mKP-I equation is generated by the nonlocal Riemann—Hilbert problem while for the mKP-II equation it is generated by the $\bar\partial$-problem. The constraints on the inverse problem data which result to real u are found. For the mKP-I equation the spectrum contains also the discrete part which corresponds to the rational nonsingular lumps decreasing at all directions. The explicit formula for the multi-lump solutions is found. The simplest, one-lump solution of the mKP-I equation is of the form

$$u(x,y,t)=4\frac{\alpha\tilde{x}^2+\beta\tilde{y}^2+2\gamma\tilde{x}\tilde{y}-\alpha c^2}{((\tilde{x}-a\tilde{y})^2+b^2\tilde{y}^2+c^2)^2+(\alpha\tilde{x}+\gamma\tilde{y})^2}, \qquad (1.5)$$

where $\tilde{x}=x-3(a^2+b^2)t+x_0$, $\tilde{y}=y-6at+y_0$ and where α, β, γ, a, b, c, x_0, y_0 are some real constants.

The scattering of the lumps is completely trivial. The explicit exact solutions of the mKP-I equation which corresponds to the degenerated inverse problem data are also found.

The use of the $\bar{\partial}$-dressing method allows us to construct very wide classes of solutions both of the mKP-I and mKP-II equation which are not necessarily decreasing at infinity and bounded. They include the rational solutions and the solutions with functional parameters. General formula for the real rational solutions of the mKP equation is

$$u(x,y,t)=-2\sigma^{-1}\frac{\partial}{\partial x}\ln\det(1+BA^{-1}), \qquad (1.6)$$

where

$$A_{ik}=\delta_{ik}\left(x+\frac{2y}{\sigma\lambda_i}+\frac{12t}{\lambda_i^2}-\frac{i\lambda_i}{2}+c_i\right)+(1-\delta_{ik})\frac{i\lambda_k^2}{\lambda_i-\lambda_k}, \quad B_{ik}=i\lambda_k,$$

where λ_i and c_i are constants. All real rational solutions of the mKP-II equation are singular. For the mKP-I equation the formula (1.6) in addition to the lumps of the type (1.5), decreasing in all directions gives also the plane bounded lumps which do not decrease in some directions. The simplest plane lump of the mKP-I equation looks like

$$u(x,y,t)=\frac{2\lambda}{\left(x-\frac{2y}{\lambda}+\frac{12t}{\lambda^2}+\delta\right)^2+\frac{\lambda^2}{4}}, \qquad (1.7)$$

where λ and δ are arbitrary real parameters. Note that the existence of the plane lumps for the mKP-I equation is a novelty. Such nonsingular plane lumps are absent in the KP, DS and Ishimori cases.

General solutions of the mKP equation with functional parameters are given by the following compact formula:

$$u(x,y,t)=2\sigma^{-1}\frac{\partial}{\partial x}\ln\frac{\det A}{\det\tilde{A}}, \qquad (1.8)$$

where

$$A_{kl}=\delta_{kl}+\frac{i}{2}\int^x dx'\,\xi_{kx'}(x',y,t)\,\eta_l(x',y,t),$$

$$\tilde{A}_{kl}=\delta_{kl}-\frac{i}{2}\int^x dx'\,\xi_k(x',y,t)\,\eta_{lx'}(x',y,t)$$

and $\xi_l(x,y,t)$, $\eta_k(x,y,t)$ are arbitrary complex valued solutions of the linearized mKP equation $u_t+u_{xxx}-3\sigma^2\partial_x^{-1}u_{yy}=0$. The solutions (1.8) are real-valued if $\xi_k=-R_k\bar{\eta}_k$ for the mKP-I equation where R_k are arbitrary real constants and if $\bar{\xi}_k=-\xi_k$, $\bar{\eta}_k=\eta_k$ for the mKP-II equation.

The class of solutions (1.8) contains as the particular cases the plane soliton solutions and breather type solutions both for the mKP-I and mKP-II equations. Explicit general formulae for such solutions are presented. Plane solitons for the mKP equation correspond to the choices $\xi_l=-R_l\bar{\eta}_l=-2iR_l\exp F(\lambda_l)$ (mKP-I) and $\xi_l=-2iR_l\exp F(i\alpha_l)$, $\eta_l=(-2/\beta_l)\exp(-F(i\beta_l))$ (mKP-II) in (1.8) where

$$F(\lambda)\stackrel{\text{def}}{=}\frac{ix}{\lambda}+\frac{y}{\sigma\lambda^2}+4i\frac{t}{\lambda^3},$$

λ_l are arbitrary complex parameters and α_l, β_l, R_l are arbitrary real parameters. They are bounded and do not decrease at some directions. General breather solutions of the mKP equation are not bounded but the singularities of some of them are integrable, i.e. $u\in L_1$. Particular exact solutions periodic in y and decaying at $x\to\pm\infty$ and periodic and moving in x and decaying in y are presented.

The Miura transformation between the mKP and KP equations is discussed. This 2+1-dimensional Miura transformation converts the real-valued solutions of the mKP-II equation into the real-valued solutions of the KP-II equation, and, in particular, it maps the plane solitons of the mKP-II equation constructed in the paper into the well-known plane solitons of the KP-II equation.

2. INITIAL-VALUE PROBLEM FOR THE mKP-I EQUATION

We start with the mKP-I equation ($\sigma=i$). It is equivalent to the compatibility condition for the linear system

$$i\Psi_y+\Psi_{xx}+iu(x,y,t)\Psi_x=0, \tag{2.1a}$$

$$\Psi_t+4\Psi_{xxx}+6iu\Psi_{xx}+\left(3iu_x+3\partial_x^{-1}u_y-\frac{3}{2}u^2\right)\Psi_x+\alpha\Psi=0. \tag{2.1b}$$

General scheme is the same as for the KP-I equation [8, 9, 17—19]. The first step is the introduction of the spectral parameter. Here the first difference between the mKP and the KP equations appears. For the problem (2.1a) it is more convenient and adequate to introduce the spectral parameter by transiting to the function χ defined by

$$\Psi=\chi(x,y,t;\lambda)\exp\left[i\left(\frac{x}{\lambda}-\frac{y}{\lambda^2}\right)\right], \tag{2.2}$$

where λ is a complex parameter. The function χ obeys the equation

$$i\chi_y+\chi_{xx}+\frac{2i}{\lambda}\chi_x+u\left(i\partial_x-\frac{1}{\lambda}\right)\chi=0. \tag{2.3}$$

It is easy to see that the function χ admits the canonical normalization $\chi\to1$ at $\lambda\to\infty$. It is also bounded at $\lambda=0$ and

$$u(x,y,t)=2i\frac{\chi_x(x,y,t;\lambda=0)}{\chi(x,y,t;\lambda=0)}. \tag{2.4}$$

Such properties of the function χ are the consequences of the definition (2.2). Alternative way of introducing the spectral parameter is discussed in Appendix A.

So, we are looking for the solutions of the linear problem (2.3) canonically normalized and bounded for all λ except, may be, the finite number of points. Such solutions of (2.3) can be found as the solutions of the linear integral equation

$$\chi(x,y,t;\lambda)=1+$$

$$+\int_{-\infty}^{+\infty}\!\!\int dx'dy'\,G(x-x',y-y';\lambda)\,u(x',y',t)\left(i\frac{\partial}{\partial x'}-\lambda\right)\chi(x',y',t;\lambda), \tag{2.5}$$

where $G(x-x',\ y-y';\lambda)$ is the Green function for the operator $L_0=i\partial_y+\partial_x^2+\frac{2i}{\lambda}\partial_x$ where $\partial_x\equiv\frac{\partial}{\partial x}$, $\partial_y\equiv\frac{\partial}{\partial y}$. The operator L_0 and the Green function G are exactly the same (up to the substitution

$\lambda\to1/\lambda$) as for the KP equation [7, 8]. Hence, similar to the KP case, one can construct the two Green functions G^+ and G^- given by the following formulae

$$G^{\pm}(x,y,t;\lambda)=\frac{i}{2\pi}\int_{-\infty}^{+\infty}\frac{d\mu}{\mu^2}\left[\Theta\left(\pm\left(\frac{1}{\mu}-\frac{1}{\lambda}\right)\right)\Theta(y)-\right.$$

$$\left.-\Theta\left(\pm\left(\frac{1}{\lambda}-\frac{1}{\mu}\right)\right)\Theta(-y)\right]\exp[\,\omega(x,y;\mu,\lambda)], \tag{2.6}$$

where $\Theta(\xi)$ is the Heaviside (step) function $\left(\Theta(\xi)=\begin{cases}1, & \xi>0\\ 0, & \xi<0\end{cases}\right)$ and

$$\omega(x,y;\mu,\lambda)\overset{def}{=}ix\left(\frac{1}{\mu}-\frac{1}{\lambda}\right)-iy\left(\frac{1}{\mu^2}-\frac{1}{\lambda^2}\right). \tag{2.7}$$

The Green functions $G^{\pm}(x,\ y;\ \lambda)$ are analytic and bounded in the upper and lower half-planes of λ, respectively.

They allow us to define the two solutions of the problem (2.3) via the integral equations

$$\chi^{\pm}(x,y;\lambda)=1+\left[G^{\pm}(\cdot;\lambda)\,u(\cdot)\left(i\partial-\frac{1}{\lambda}\right)\chi^{\pm}(\cdot;\lambda)\right](x,y). \tag{2.8}$$

As far as the Green functions G^+ and G^- the solutions χ^+ and χ^- of equations (2.8) are bounded and analytic in the upper and lower half-planes of λ, respectively, except the points λ_i where the homogeneous equations (2.8) have nontrivial solutions. Further, since $G^+-G^-\neq0$ at $\mathrm{Im}\,\lambda=0$, then $\chi^+-\chi^-\neq0$ at $\mathrm{Im}\,\lambda=0$ too. Then, one can define the function

$$\chi=\begin{cases}\chi^+, & \mathrm{Im}\,\lambda>0\\ \chi^-, & \mathrm{Im}\,\lambda<0\end{cases},$$

which is analytic and bounded on the entire complex plane (except finite number of points) and has a jump across the real axis.

So, we arrive at the standard singular Riemann—Hilbert problem. The nature of this problem can be analyzed exactly in the same manner as for the KP equation [7, 9]. To do this one have to express the jump $\Delta(x,y,t;\lambda)\overset{def}{=}\chi^+(x,y,t;\lambda)-\chi^-(x,y,t;\lambda)$ at $\mathrm{Im}\,\lambda=0$ via χ^-.

Subtraction of equations (2.8) gives

$$\chi^+(\lambda)-\chi^-(\lambda)=(G^+(\lambda)-G^-(\lambda))\,u\Big(i\partial-\frac{1}{\lambda}\Big)\chi^+(\lambda)+$$

$$+G^-(\lambda)\,u\Big(i\partial-\frac{1}{\lambda}\Big)(\chi^+(\lambda)-\chi^-(\lambda)). \qquad (2.9)$$

Using the expression for $G^+(\lambda)-G^-(\lambda)$ at $\mathrm{Im}\,\lambda=0$

$$(G^+-G^-)(x,y;\lambda)=\frac{i}{2\pi}\int_{-\infty}^{+\infty}\frac{d\mu}{\mu^2}\,\mathrm{Sgn}\Big(\frac{1}{\mu}-\frac{1}{\lambda}\Big)\exp[\,\omega(x,y;\mu,\lambda)], \qquad (2.10)$$

which follows from (2.6) and easily verified identity

$$\Big(i\partial_\xi-\frac{1}{\lambda}\Big)(\Phi(\xi,\eta)\exp[\,\omega(\xi,\eta;\mu,\lambda)])=$$

$$=\exp[\,\omega(\xi,\eta;\mu,\lambda)]\,\Big(i\partial_\xi-\frac{1}{\mu}\Big)\Phi(\xi,\eta), \qquad (2.11)$$

one gets

$$\Delta(x,y;\lambda)=\int_{-\infty}^{+\infty}\frac{d\mu}{\mu^2}T(\lambda,\mu)\exp[\,\omega(x,y;\mu,\lambda)]+$$

$$+\Big[G^-(\cdot;\lambda)\,u(\cdot)\Big(i\partial-\frac{1}{\lambda}\Big)\Delta(\cdot;\lambda)\Big](x,y), \qquad (2.12)$$

where

$$T(\lambda,\mu)=\frac{i}{2\pi}\mathrm{Sgn}\Big(\frac{1}{\mu}-\frac{1}{\lambda}\Big)\times$$

$$\times\int\!\!\int_{-\infty}^{+\infty}d\xi\,d\eta\,u\Big(i\partial_\xi-\frac{1}{\mu}\Big)\chi^+(\xi,\eta)\exp[\,\omega(\xi,\eta;\lambda,\mu)]. \qquad (2.13)$$

Then we introduce the solution $N(x,\ y;\ \mu,\ \lambda)$ of the problem (2.3) which is also the solution of the integral equation

$$N(x,y;\mu,\lambda)=\exp[\,\omega(x,y;\mu,\lambda)]+$$

$$+\Big[G^+(\cdot;\lambda)\,u(\cdot)\Big(i\partial-\frac{1}{\lambda}\Big)N(\cdot;\mu,\lambda)\Big](x,y), \qquad (2.14)$$

where ω is given by (2.7). Comparing the integral equations (2.12) and (2.14) and assuming that the homogeneous equation (2.8) (and (2.12)) has no nontrivial solution for real λ, one concludes

$$\Delta(x,y;\lambda)=\int_{-\infty}^{+\infty}\frac{d\mu}{\mu^2}T(\lambda,\mu)\,N(x,y;\mu,\lambda). \qquad (2.15)$$

Now one needs to express N via χ^-. It follows from (2.8) and (2.14) that

$$\chi^-(x,y;\lambda)\exp[\,\tilde\omega(x,y;\lambda)]-$$

$$-\Big[\tilde G^-(\cdot;\lambda)\,u(\cdot)\Big(i\partial-\frac{1}{\lambda}\Big)\chi^-(\cdot;\lambda)\exp[\,\tilde\omega(\cdot;\lambda)]\Big](x,y)=\exp[\,\tilde\omega(x,y;\lambda)], \qquad (2.16)$$

$$\frac{\partial}{\partial(\lambda^{-1})}(N(x,y;\mu,\lambda)\exp[\,\tilde\omega(x,y;\lambda)])-$$

$$-\Big[\tilde G^-(\cdot;\lambda)\,u(\cdot)\Big(i\partial-\frac{1}{\lambda}\Big)\frac{\partial(N(\cdot;\mu,\lambda)\exp[\,\tilde\omega(\cdot;\lambda)])}{\partial(\lambda^{-1})}\Big](x,y)=$$

$$=\exp[\,\tilde\omega(x,y;\lambda)]\,F(\mu,\lambda), \qquad (2.17)$$

where $\tilde\omega(x,y;\lambda)=i\Big(\frac{x}{\lambda}-\frac{y}{\lambda^2}\Big)$, the Green function $\tilde G^-$ acts as

$$(\tilde G^-\Phi)(x,y)=\exp[\,\tilde\omega(x,y;\lambda)]\,(G^-(\cdot;\lambda)\exp[-\tilde\omega(\cdot;\lambda)]\,\Phi(\cdot))(x,y) \qquad (2.18)$$

and

$$F(\mu,\lambda)=\frac{i}{2\pi}\int\!\!\int_{-\infty}^{+\infty}d\xi\,d\eta\,u(\xi,\eta)\Big(i\partial_\xi-\frac{1}{\lambda}\Big)N(\xi,\eta;\mu,\lambda). \qquad (2.19)$$

The comparison of equations (2.16) and (2.17) gives

$$\frac{\partial}{\partial(\lambda^{-1})}(N(x,y;\mu,\lambda)\exp[\,\tilde\omega(x,y;\lambda)])=F(\mu,\lambda)\,\chi^-(x,y;\lambda)\exp[\,\tilde\omega(x,y;\lambda)]. \qquad (2.20)$$

Since $N(x,\ y;\ \lambda,\ \lambda)=\chi^-(x,\ y;\ \lambda)$ at $\mathrm{Im}\,\lambda=0$, equation (2.20) implies

$$N(x,y;\mu,\lambda)=\chi^-(x,y;\mu)\exp[\,\omega(x,y;\mu,\lambda)]-$$

$$-\int_\mu^\lambda\frac{d\rho}{\rho^2}F(\mu,\rho)\,\chi^-(x,y;\rho)\exp[\,\omega(x,y;\rho,\lambda)]. \qquad (2.21)$$

Substituting (2.21) into (2.15), one obtains the expression for the jump $\chi^+-\chi^-$ via χ^- we are interesting in:

$$\chi^+(x,y;\lambda)-\chi^-(x,y;\lambda)=\int_{-\infty}^{+\infty}\frac{d\mu}{\mu^2}T(\lambda,\mu)\,\chi^-(x,y;\mu)\exp[\,\omega(x,y;\mu,\lambda)]-$$

$$-\int_{-\infty}^{+\infty}\frac{d\mu}{\mu^2}T(\lambda,\mu)\int_{\mu}^{\lambda}\frac{d\rho}{\rho^2}F(\mu,\rho)\,\chi^-(x,y;\rho)\exp[\,\omega(x,y;\rho,\lambda)], \qquad (2.22)$$

where the functions $T(\lambda,\ \mu)$ and $F(\mu,\ \lambda)$ are given by (2.13) and (2.19) respectively. Changing the order of integration in the r.h.s. of (2.22), one finally gets

$$\chi^+(x,y;\lambda)-\chi^-(x,y;\lambda)=\int_{-\infty}^{+\infty}\frac{d\mu}{\mu^2}f(\lambda,\mu)\,\chi^-(x,y;\mu)\exp[\,\omega(x,y;\mu,\lambda)], \qquad (2.23)$$

$$(\mathrm{Im}\,\lambda=0)$$

where

$$f(\lambda,\mu)=T(\lambda,\mu)-\Theta(\lambda-\mu)\int_{-\infty}^{+\infty}\frac{d\rho}{\rho^2}T(\lambda,\rho)\,F(\rho,\mu)+$$

$$+\Theta(\mu-\lambda)\int_{\mu}^{+\infty}\frac{d\rho}{\rho^2}T(\lambda,\rho)\,F(\rho,\mu) \qquad (2.24)$$

and $\omega(x,\ y;\ \mu,\ \lambda)$ is given by (2.7).

The relation (2.23) demonstrates that the Riemann—Hilbert problem we are dealing with is the nonlocal Riemann—Hilbert problem. Similar to the KP equation the nonlocal Riemann—Hilbert problem (2.23) generates the inverse problem equations for the problem (2.3)

To obtain the complete set of the inverse problem equations one has to take into account the possible singularities of the functions χ^+ and χ^-. In our case equations (2.3) and (2.8) are not self-adjoint. As a result, the structure of singularities for the functions χ^+ and χ^- may be rather complicated. Here we will restrict ourselves by the consideration of the simplest, the simple pole singularities, i.e. we will consider the functions χ^+ and χ^- of the form

$$\chi^{\pm}(x,y;\lambda)=1+\sum_{k=1}^{n_\pm}\frac{c_k^{\pm}\chi_k^{\pm}(x,y)}{\lambda-\lambda_k^{\pm}}+\tilde{\chi}^{\pm}(x,y;\lambda)\,, \qquad (2.25)$$

where $\tilde{\chi}^+$ and $\tilde{\chi}^-$ are the functions analytic in the upper and lower half planes, $c_k^{\pm}$ are normalization constants and as usual $\chi_k^{\pm}(x,\ y)$ are the solutions of the homogeneous equations (2.8)

$$\chi_k^{\pm}(x,y)=\left[G^{\pm}(\cdot\,;\lambda_k^{\pm})\,u(\cdot)\left(i\partial-\frac{1}{\lambda_k^{\pm}}\right)\chi_k^{\pm}(\cdot)\right](x,y), \qquad k=1,\dots,n_{\pm}. \qquad (2.26)$$

We will normalize the functions $\chi_k^{\pm}(x,\ y)$ as follows

$$\lim_{(x^2+y^2)^{1/2}\to\infty}\left(x-\frac{2y}{\lambda_k^{\pm}}\right)\chi_k^{\pm}(x,y)=1\,. \qquad (2.27)$$

In such normalization one has $c_k^{\pm}=-i\lambda_k^{\pm 2}$ (see Appendix B). For the functions $\chi_k^{\pm}(x,\ y)$ the following important relations are also hold:

$$\lim_{\lambda\to\lambda_k^{\pm}}\left(\chi^{\pm}(x,y;\lambda)-\frac{(-i\lambda_k^{\pm 2})\,\chi_k^{\pm}(x,y)}{\lambda-\lambda_k^{\pm}}\right)=\left(x-\frac{2y}{\lambda_k^{\pm}}+\gamma_k^{\pm}\right)\chi_k^{\pm}(x,y)\,, \qquad (2.28)$$

where $\gamma_k^{\pm}(t)$ are some independent on x and y functions. The derivation of (2.28) is similar to that in the KP case. We will present it for completeness in the Appendix B.

Thus, we have the nonlocal Riemann—Hilbert problem with the simple poles. Using (2.25), (2.23) and standard formulae from the theory of complex variables, one gets

$$\chi(x,y;\lambda)=1+\sum_{k=1}^{n_+}\frac{(-i\lambda_k^{+2})\,\chi_k^+(x,y)}{\lambda-\lambda_k^+}+\sum_{k=1}^{n_-}\frac{(-i\lambda_k^{-2})\,\chi_k^-(x,y)}{\lambda-\lambda_k^-}+$$

$$+\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{d\mu}{\mu-\lambda}\int_{-\infty}^{+\infty}\frac{d\rho}{\rho^2}f(\mu,\rho)\,\chi^-(x,y;\rho)\exp[\,\omega(x,y;\rho,\mu)]. \qquad (2.29)$$

Proceeding in (2.29) to the limits $\mathrm{Im}\,\lambda\to-0$, $\lambda\to\lambda_k^+$, $\lambda\to\lambda_k^-$, and using (2.28), we obtain the system of equations

$$\chi^-(x,y;\lambda)+i\sum_{k=1}^{n_+}\frac{\lambda_k^{+2}\chi_k^+}{\lambda-\lambda_k^+}+i\sum_{k=1}^{n_-}\frac{\lambda_k^{-2}\chi_k^-}{\lambda-\lambda_k^-}-$$

$$-\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{d\mu}{\mu-(\lambda-i0)}\int_{-\infty}^{+\infty}\frac{d\rho}{\rho^2}f(\mu,\rho)\,\chi^-(x,y;\rho)\exp[\,\omega(x,y;\rho,\mu)]=1\,, \qquad (2.30)$$

$$(\mathrm{Im}\,\lambda=0)$$

$$\left(x-\frac{2y}{\lambda_j^\pm}+\gamma_j^\pm\right)\chi_j^\pm+i\sum_{\substack{k=1\\k\neq j}}^{n_+}\frac{\lambda_k^{+2}\chi_k^+}{\lambda_j^\pm-\lambda_k^+}+$$

$$+i\sum_{k=1}^{n_-}\frac{\lambda_k^{-2}\chi_k^-}{\lambda_j^\pm-\lambda_k^-}-\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{d\mu}{\mu-\lambda_j^\pm}\int_{-\infty}^{+\infty}\frac{d\rho}{\rho^2}\times$$

$$\times f(\mu,\rho)\,\chi^-(\rho)\exp\left[\omega(x,y;\rho,\mu)\right]=1\,,\quad j=1,\ldots,n_\pm\,. \tag{2.31}$$

Together with the reconstruction formula for «potential» u

$$u(x,y,t)=2i(\ln\chi_0)_x\,, \tag{2.32}$$

where

$$\chi_0(x,y)\overset{def}{=}\chi(x,y;\lambda=0)=1+i\sum_{k=1}^{n_+}\lambda_k^+\chi_k^++i\sum_{k=1}^{n_-}\lambda_k^-\chi_k^-+$$

$$+\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{d\mu}{\mu}\int_{-\infty}^{+\infty}\frac{d\rho}{\rho^2}f(\mu,\rho)\,\chi^-(\rho)\exp\left[\omega(x,y;\rho,\mu)\right] \tag{2.33}$$

the system of equations (2.30), (2.31) form the inverse problem equations for the problem (2.3). The set $\mathcal{F}\overset{def}{=}\{f(\mu,\rho),\ -\infty<\mu,\rho<+\infty\,;\ \lambda_k^+,\ \gamma_k^+\ (k=1,\ldots,n_k^+)\,;\ \lambda_k^-,\gamma_k^-\ (k=1,\ldots,n_-)\}$ is the inverse problem data. The inverse problem equations (2.30), (2.31) are uniquelly solvable at least for small data. Solving equations (2.30), (2.31) for given $\mathcal{F}$, one reconstructs the potential $u(x,y)$ via (2.32). Note that this set of inverse data is not complete, since we have restricted only by the simple pole contribution to the discrete spectrum. Multiple poles will be considered elsewhere.

Emphasize one important point. The solution $\chi(\lambda)$ of the inverse problem equations (2.30), (2.31) obeys the linear equation (2.3) with u given by (2.32) for any data $\mathcal{F}$ without any constraint. This can be easily seen by the substitution of the asymptotic expansion $\chi=1+\frac{1}{\lambda}\chi_{-1}+\frac{1}{\lambda^2}\chi_{-2}+\ldots$ at $\lambda\to\infty$ into (2.3). Such a property of the inverse problem equations and data is the important advantage of the way (2.2) of the introduction of the spectral parameter.

To apply the inverse problem equations obtained for the linearization of the initial-value problem for the mKP-I equation one has to find, as usual, the time dependence of the inverse problem data.

This can be done in a standard manner by consideration of the linear equation (2.1b) with $\int_{-\infty}^{+\infty}dx\,u(x,y,0)=0$ and $\alpha=4i\lambda^{-3}$ at the limit $x^2+y^2\to\infty$. One gets

$$\frac{\partial f(\lambda,\mu,t)}{\partial t}=4i\left(\frac{1}{\mu^3}-\frac{1}{\lambda^3}\right)f(\lambda,\mu,t)\,,$$

$$\frac{\partial\lambda_k^\pm}{\partial t}=0\,,\qquad\frac{\partial\gamma_k^\pm(t)}{\partial t}=\frac{12}{\lambda_k^{\pm2}}\,,\qquad k=1,\ldots,n_\pm\,. \tag{2.34}$$

Hence

$$f(\lambda,\mu,t)=f(\lambda,\mu,0)\exp\left[4i\left(\frac{1}{\mu^3}-\frac{1}{\lambda^3}\right)t\right],$$

$$\gamma_k^\pm(t)=\frac{12t}{\lambda_k^{\pm2}}+\gamma_k^\pm(0)\,,\qquad\lambda_k^\pm(t)=\lambda_k^\pm(0)\,, \tag{2.35}$$

where $f(\lambda,\mu,0)$ and $\gamma_k^\pm(0)$ are arbitrary function and constants respectively.

The formulae (2.35) and the inverse problem equations (2.30) — (2.32) allow us to solve the initial value problem for the mKP-I equation by the standard IST procedure [1—6]

$$u(x,y,0)\to\mathcal{F}(0)\to\mathcal{F}(t)\to u(x,y,t)\,. \tag{2.36}$$

The solution $u(x,y,t)$ of the mKP-I equation reconstructed from the generic inverse problem data $\mathcal{F}$ is the complex-valued one in general. Of course, the reduction to the real valued u is of the main interest. Unfortunately we are not able to describe such a reduction as an involution for the function χ. The constraints on the inverse problem data $\mathcal{F}$ which quarantee the reality of u can be obtained in a different manner, namely, by the analysis of the weak u limit of the inverse problem equations. Indeed, for small (in a suitable sence) $u(x,y,t)$ one has

$$\chi^\pm\sim1\,,\qquad N\sim\exp\left[\omega(x,y;\mu,\lambda)\right]\,,$$

$$F(\mu,\lambda)\simeq-\frac{i}{2\pi\mu}\iint_{-\infty}^{+\infty}d\xi\,d\eta\,u(\xi,\eta,t)\exp\left[\omega(\xi,\eta;\mu,\lambda)\right]\,. \tag{2.37}$$

Hence for small u

$$f(\lambda,\mu) \simeq \frac{i}{2\pi\lambda} \operatorname{Sgn}\left(\frac{1}{\lambda}-\frac{1}{\mu}\right) \times$$

$$\times \int\int_{-\infty}^{+\infty} d\xi\, d\eta\, u(\xi,\eta) \exp\left[i\xi\left(\frac{1}{\lambda}-\frac{1}{\mu}\right)-i\eta\left(\frac{1}{\lambda^2}-\frac{1}{\mu^2}\right)\right]. \qquad (2.38)$$

Now it is easy to see that the condition $\bar{u}=u$, where bar means the complex conjugation, implies

$$\overline{f(\mu,\lambda)}\ \mu = f(\lambda,\mu)\ \lambda, \qquad \operatorname{Im}\lambda = \operatorname{Im}\mu = 0. \qquad (2.39)$$

It is not difficult to show, using (2.28), (2.29) and (2.5), that the constraint (2.29) guarantees the reality of u for not small u too. The reality constraints can be derived also for the discrete part of the inverse problem data: $n_+=n_-$, $\lambda_k^+=\overline{\lambda_k^-}$, $\gamma_k^\pm=-i\lambda_k^\pm/2+\delta_k^\pm$, $\delta_k^+=\overline{\delta_k^-}$.

3. EXACT SOLUTIONS OF THE mKP-I EQUATION

Exact solutions of the mKP-I equation can be found explicitly, as usual, for the pure discrete data and for the degenerated pure continuous data.

For the pure discrete inverse problem data ($f(\lambda,\mu)\equiv 0$) the inverse problem equations (2.30) — (2.32) are reduced to the following

$$\left(x-\frac{2y}{\lambda_j^\pm}+\frac{12t}{\lambda_j^{\pm 2}}+\gamma_j^\pm(0)\right)\chi_j^\pm + i\sum_{\substack{k=1\\k\neq j}}^{n_+}\frac{\lambda_k^{+2}\chi_k^+}{\lambda_j^\pm-\lambda_k^+}+$$

$$+i\sum_{\substack{k=1\\k\neq j}}^{n_-}\frac{\lambda_k^{-2}\chi_k^-}{\lambda_j^\pm-\lambda_k^-}=1, \qquad j=1,\ldots,n_\pm \qquad (3.1)$$

and

$$u(x,y,t)=2i\frac{\partial}{\partial x}\ln\left(1+i\sum_{k=1}^{n_+}\lambda_k^+\chi_k^+ + i\sum_{k=1}^{n_-}\lambda_k^-\chi_k^-\right). \qquad (3.2)$$

The algebraic system (3.1) can be readily solved and then (3.2) gives us the explicit solutions of the mKP-I equation. They are the rational function on x, y and t. The real-valued $u(x,y,t)$ arises only for the special choice of n_+, n_-, λ_k^+,λ_k^- and $\gamma_k^\pm$. It is not difficult to show that the function $u(x,y,t)$ is real one if

$$n_+=n_-\equiv n, \qquad \lambda_k^-=\overline{\lambda_k^+}, \qquad \gamma_k^\pm=-\frac{i}{2}\lambda_k^\pm+\delta_k^\pm, \qquad (3.3)$$

where $\delta_k^-=\overline{\delta_k^+}$ are arbitrary constants.

In this case introducing the notations

$$(\lambda_1^+,\ldots,\lambda_n^+,\lambda_1^-,\ldots,\lambda_n^-)\equiv(\lambda_1,\ldots,\lambda_{2n}),$$

$$(\gamma_1^+(0),\ldots,\gamma_n^+(0),\gamma_1^-(0),\ldots,\gamma_n^-(0))\equiv(\gamma_1(0),\ldots,\gamma_{2n}(0)), \qquad (3.4)$$

$$(\chi_1^+,\ldots,\chi_n^+,\chi_1^-,\ldots,\chi_n^-)\equiv(\chi_1,\ldots,\chi_{2n}),$$

one can rewrite the system (3.1) in a more compact form

$$\left(x-\frac{2y}{\lambda_j}+\frac{12t}{\lambda_j^2}+\gamma_j(0)\right)\chi_j+\sum_{\substack{k=1\\k\neq 1}}^{2n}\frac{i\lambda_k^2\chi_k}{\lambda_j-\lambda_\kappa}=1 \quad (j=1,\ldots,2n). \qquad (3.5)$$

Solving (3.5), one obtains

$$u(x,y,t)=2i\frac{\partial}{\partial x}\ln\left(\frac{\det A+i\sum_{k,j}\lambda_k\tilde{A}_{kj}}{\det A}\right) \qquad (3.6)$$

where

$$A_{jk}=\left(x-\frac{2y}{\lambda_j}+\frac{12t}{\lambda_j^2}+\gamma_j(0)\right)\delta_{jk}+(1-\delta_{jk})\frac{i\lambda_k^2}{\lambda_j-\lambda_k}, \quad j,k=1,\ldots,2n \quad (3.7)$$

and $\tilde{A}_{kj}$ is the cofactor of the element A_{jk}. In virtue of (3.3), one has

$$\det A+i\sum_{k,j}\lambda_k\tilde{A}_{kj}=\overline{\det A}. \qquad (3.8)$$

Hence

$$u(x,y,t)=4\frac{\partial}{\partial x}\operatorname{Im}\ \ln\det A=4\frac{\partial}{\partial x}\operatorname{arctg}\frac{\operatorname{Im}\det A}{\operatorname{Re}\det A}, \qquad (3.9)$$

where A is given by (3.7). One can also show that under the constraint (3.3) the function $u(x, y, t)$ is bounded on the whole plane (x, y). So, the formula (3.9) gives us the rational nonsingular solutions, i.e. the lumps of the mKP-I equation. The simplest, one-lump solution is of the form $(n=1)$

$$u(x, y, t)=4\frac{\lambda_R\tilde{x}^2+4|\lambda|^2\,\mathrm{Re}(1/\lambda^3)\,\tilde{y}^2-4|\lambda|^2\,\mathrm{Re}(1/\lambda^2)\tilde{x}\tilde{y}-\frac{\lambda_R^3}{4\lambda_I^2}|\lambda|^4}{\left(\left(\tilde{x}-\frac{2\lambda_R}{|\lambda|^2}\tilde{y}\right)^2+\frac{4\lambda_I^2}{|\lambda|^4}\tilde{y}^2+\frac{\lambda_R^2|\lambda|^4}{4\lambda_I^2}\right)^2+(\lambda_R\tilde{x}-2|\lambda|^2\,\mathrm{Re}(1/\lambda^2)\,\tilde{y})^2} \tag{3.10}$$

where

$$\tilde{x}=x-\frac{12t}{|\lambda|^2}+\delta_R+\delta_I\frac{\lambda_R}{\lambda_I}, \qquad \tilde{y}=y-\frac{12\lambda_R t}{|\lambda|^2}+\delta_I\frac{|\lambda|^2}{2\lambda_I} \tag{3.11}$$

and $\delta^+=\delta_R+i\delta_I$.

The lump (3.10) moves with the velocity $V=(V_x, V_y)=\left(\frac{12}{|\lambda|^2}, \frac{12\lambda_R}{|\lambda|^2}\right)$ and decays as $(x^2+y^2)^{-1}$ in all directions on the plane (x, y).

The general solution (3.9) describes the scattering of n lumps of the form (3.10). It follows from (3.9) and (3.7) that at $t\to\pm\infty$:

$$u(x, y, t)=\sum_{\alpha=1}^{n} u_\alpha(x-V_{\alpha x}t,\ y-V_{\alpha y}t), \tag{3.12}$$

where u_α are the one-lump solutions (3.10). Hence, thee scattering of the lumps is completely trivial similar to the KP-I case: the phase shift is absent.

The lump solutions (3.9) are the transparent «potentials» for the problem (2.1a).

Another class of explicit solutions corresponds to the degenerated pure continuous data, i.e.

$$\lambda_k^{\pm}=0, \qquad \gamma_k^{\pm}=0$$

and

$$f(\lambda, \mu, 0)=\sum_{k=1}^{n} g_k(\lambda)\, h_k(\mu) \tag{3.13}$$

where g_k and h_k are arbitrary functions.

In this case the inverse problem equations, i.e. the equation

$$\chi^-(x, y, \lambda)-\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{d\mu}{\mu-(\lambda-i0)}\int_{-\infty}^{+\infty}\frac{d\rho}{\rho^2}f(\mu, \rho)\,\chi^-(\rho)\,e^{\omega(x, y;\rho, \mu)}=1 \tag{3.14}$$

is reduced also to the algebraic system.

Here we will consider directly the real-valued solutions $u(x, y, t)$. For the functions $f(\lambda, \mu, 0)$ of the form (3.13) the reality condition (2.39) is satisfied if

$$h_k(\mu)=R_k\,\overline{g_k(\mu)}\ \ \mu. \tag{3.15}$$

Where R_k are arbitrary real constants. So

$$f(\lambda, \mu, 0)=\sum_{k=1}^{n} R_k\, g_k(\lambda)\,\overline{g_k(\mu)}\ \ \mu. \tag{3.16}$$

Substitution of the function $f(\lambda, \mu, 0)$ of the form (3.16) into (3.14) give rise to the algebraic system

$$\sum_{k=1}^{n} A_{lk}\xi_k=h_l, \qquad l=1, \dots, n, \tag{3.17}$$

where

$$h_k(x, y, t)\overset{def}{=}\frac{R_k}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{d\lambda}{\lambda^2}\lambda\,\overline{g_k(\lambda)}\ \exp\left[i\left(\frac{x}{\lambda}-\frac{y}{\lambda^2}+\frac{4t}{\lambda^3}\right)\right], \tag{3.18}$$

$$\xi_k(x, y, t)\overset{def}{=}\frac{R_k}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{d\lambda}{\lambda^2}\lambda\,\overline{g_k(\lambda)}\ \chi^-(\lambda)\exp\left[i\left(\frac{x}{\lambda}-\frac{y}{\lambda^2}+\frac{4t}{\lambda^3}\right)\right], \tag{3.19}$$

$$A_{lk}(x, y, t)\overset{def}{=}\delta_{lk}+\frac{iR_l}{2\pi}\int_{-\infty}^{+\infty}\frac{d\lambda}{\lambda^2}\overline{g_l(\lambda)}\ \exp\left[i\left(\frac{x}{\lambda}-\frac{y}{\lambda^2}+\frac{4t}{\lambda^3}\right)\right]\times$$

$$\times\int_{-\infty}^{+\infty}\frac{d\mu}{\mu-(\lambda-i0)}g_k(\mu)\ \exp\left[-i\left(\frac{x}{\mu}-\frac{y}{\mu^2}+\frac{4t}{\mu^3}\right)\right]. \tag{3.20}$$

For given arbitrary functions $g_k(\lambda)$ $(k=1, ..., n)$, solving the system (3.17)

$$\xi_l=\sum_k (A^{-1})_{lk} h_k \tag{3.21}$$

and taking into account (2.29), one finds

$$\chi^-(\lambda=0)=1-i\sum_{l,k} R_l^{-1} \bar{h}_l (A^{-1})_{lk} h_k=1+\text{tr}\, M \tag{3.22}$$

where

$$M_{lk} \stackrel{def}{=} -i R_l^{-1} \bar{h}_l (A^{-1})_{lk} h_k . \tag{3.23}$$

Matrix M has rank one. Hence, $\det(1+M)=1+\text{tr}\, M$ and $\chi^-(0)=\det(1+M)$. Then using the identity

$$\frac{1}{\lambda(\mu-(\lambda-i0))}=\frac{1}{\lambda\mu}-\frac{1}{\mu(\lambda-(\mu+i0))}, \tag{3.24}$$

one can show that

$$R_l A_{lk}(x,y,t)-(A^+)_{lk}(x,y,t) R_k=ih_l(x,y,t)\, \bar{h}_k(x,y,t) . \tag{3.25}$$

As a result one has

$$\chi_0=\det(1+M)=\det(A^+A^{-1})=\frac{\overline{\det A}}{\det A}. \tag{3.26}$$

Thus, the solutions of the mKP-I equation which correspond to the data (3.16) are representable in the form

$$u(x,y,t)=4\frac{\partial}{\partial x}\text{arctg}\frac{\text{Im}\det A}{\text{Re}\det A}, \tag{3.27}$$

where matrix A is given (3.20). Solutions (3.27) depend on the n arbitrary complex-valued functions of one variables.

The solutions (3.27) can be rewritten also in the terms of the functions h_k only. Indeed, using the identity

$$\frac{i}{\mu-\lambda+i0}=\frac{1}{\lambda\mu}\int_{-\infty}^{0} dx\, \exp\left[-i\left(\frac{1}{\lambda}-\frac{1}{\mu}-i0\right)x\right], \tag{3.28}$$

one gets

$$A_{lk}(x,y,t)=\delta_{lk}+iR_k^{-1}\int_{+\infty}^{x} dx' \frac{\partial h_l(x',y,t)}{\partial x'} \bar{h}_k(x',y,t) . \tag{3.29}$$

The functions $h_k(x, y, t)$ are evidently the solutions of the linearized mKP-I equation

$$(h_t+h_{xxx})_x-3h_{yy}=0 . \tag{3.30}$$

So, the solutions (3.27) of the mKP-I equation are parametrized by the n arbitrary complex ($2n$ real-valued) solutions of its linearization (3.30) of the form (3.18) decreasing at infinity.

The simplest solution of this type is

$$u(x,y,t)=4\frac{\partial}{\partial x}\text{arctg}\frac{|h(x,y,t)|^2/2}{R+\text{Re}\left(i\int_{+\infty}^{x} dx'\, h_{x'}(x',y,t)\,\bar{h}(x',y,t)\right)}, \tag{3.31}$$

where $h(x, y, t)$ is an arbitrary solution of equation (3.30) of the form (3.18) decreasing at infinity.

4. INITIAL VALUE PROBLEM FOR THE mKP-II EQUATION

For the mKP-II equation (1.3) $(\sigma=1)$ the corresponding linear problem is of the form

$$\Psi_y+\Psi_{xx}+u\Psi_x=0 . \tag{4.1}$$

Similar to the KP case [8] the properties of the solutions of the problem (4.1) are cardinally different from those for the mKP-I equation.

Spectral parameter λ is introduced similar to the mKP-I case:

$$\Psi(x,y,t)=\chi(x,y,t;\lambda)\exp\left[i\frac{x}{\lambda}+\frac{y}{\lambda^2}\right]. \tag{4.2}$$

The function $\chi(x, y, t; \lambda)$ (4.2) obeys the equation

$$\chi_y+\chi_{xx}+\frac{2i}{\lambda}\chi_x+u\left(\partial_x+\frac{i}{\lambda}\right)\chi=0 . \tag{4.3}$$

The solutions of (4.3) admit the canonical normalization

$$\chi \underset{\lambda\to\infty}{\longrightarrow} 1+\frac{\chi_{-1}}{\lambda}+\dots \tag{4.4}$$

and

$$u(x, y, t) = -2\frac{\partial}{\partial x}\ln\chi_0\,, \tag{4.5}$$

where $\chi_0 \overset{def}{=} \chi(x, y; \lambda=0)$.

So, we will looking for the solutions of the problem (4.3) bounded on the whole complex plane λ and normalized as $\chi \underset{\lambda\to\infty}{\longrightarrow} 1$. Such solutions can be constructed as the solutions of the integral equation

$$\chi(x, y; \lambda) = 1 + \left[G(\cdot\,; \lambda)u\left(\partial + \frac{i}{\lambda}\right)\chi(\cdot)\right](x, y) \tag{4.6}$$

where $G(x, y; \lambda)$ is the bounded Green function of the operator $L_0 = \partial_y + \partial_x^2 + (2i/\lambda)\,\partial_x$. It is exactly the same Green function (up to the change $\lambda \to \lambda^{-1}$) as for the KP-II equation [9]:

$$G(x, y; \lambda) =$$

$$= \frac{1}{2\pi}\left\{\Theta(\lambda_R)\left[\Theta(-y)\left(\int_0^\infty \frac{d\mu}{\mu^2} + \int_{-|\lambda|^2/2\lambda_R}^{0} \frac{d\mu}{\mu^2}\right) - \Theta(y)\int_{-\infty}^{-|\lambda|^2/2\lambda_R} \frac{d\mu}{\mu^2}\right] + \right.$$

$$\left. + \Theta(-\lambda_R)\left[\Theta(-y)\left(\int_{-\infty}^{0} \frac{d\mu}{\mu^2} + \int_0^{-|\lambda|^2/2\lambda_R} \frac{d\mu}{\mu^2}\right) - \Theta(y)\int_{-|\lambda|^2/2\lambda_R}^{+\infty} \frac{d\mu}{\mu^2}\right]\right\} \times$$

$$\times \exp\left(i\frac{x}{\mu} + \frac{y}{\mu^2} + \frac{2y}{\lambda\mu}\right). \tag{4.7}$$

This Green function is non-analytic everywhere:

$$\frac{\partial G(x, y; \lambda)}{\partial\bar\lambda} = \frac{1}{2\pi}\frac{\lambda^2}{|\lambda|^4}\operatorname{Sgn}(\lambda_R)\exp(ipx + iqy)\,, \tag{4.8}$$

where

$$\frac{\partial}{\partial\bar\lambda} \overset{def}{=} \frac{1}{2}\left(\frac{\partial}{\partial\lambda_R} + i\frac{\partial}{\partial\lambda_I}\right)$$

and

$$p = \frac{\lambda + \bar\lambda}{|\lambda|^2}\,, \qquad q = i\frac{\bar\lambda^2 - \lambda^2}{|\lambda|^4}\,. \tag{4.9}$$

Hence, the solutions of equation (4.6) is non-analytic everywhere too. Following to the standard $\bar\partial$-method, one has to calculate now $\partial\chi/\partial\bar\lambda$. Using (4.8), one gets

$$\frac{\partial\chi(x, y; \lambda, \bar\lambda)}{\partial\bar\lambda} = F(\lambda, \bar\lambda)\exp(ipx + iqy) + \left[G(\cdot\,; \lambda, \bar\lambda)u(\cdot)\left(\partial + \frac{i}{\lambda}\right)\frac{\partial\chi}{\partial\bar\lambda}\right](x, y) \tag{4.10}$$

where

$$F(\lambda, \bar\lambda) = \frac{\lambda^2}{2\pi|\lambda|^4}\operatorname{Sgn}(\lambda_R) \times$$

$$\times \int\int_{-\infty}^{+\infty} d\xi\, d\eta\, \exp(-ip\xi - iq\eta)\, u(\xi, \eta)\left(\partial_\xi + \frac{i}{\lambda}\right)\chi(\xi, \eta; \lambda, \bar\lambda)\,.$$

Introducing the function $N(x, y; \lambda, \bar\lambda)$ which obeys the integral equation

$$N(x, y; \lambda, \bar\lambda) = \exp(ipx + iqy) + \left[G(\cdot\,; \lambda, \bar\lambda)\, u(\cdot)\left(\partial + \frac{i}{\lambda}\right)N(\cdot\,; \lambda, \bar\lambda)\right](x, y) \tag{4.12}$$

and assuming that homogeneous equation (4.6) has no nontrivial solutions, one gets

$$\frac{\partial\chi}{\partial\bar\lambda} = F(\lambda, \bar\lambda)\, N(x, y; \lambda, \bar\lambda)\,. \tag{4.13}$$

The interrelation between $N(x, y; \lambda, \bar\lambda)$ and $\chi(x, y; \lambda\ \bar\lambda)$ follows from (4.12), (4.6) and the symmetry property of the Green function (4.7)

$$G(x, y; \lambda, \bar\lambda) = G(x, y; -\bar\lambda, -\lambda)\exp(ipx + iqy) \tag{4.14}$$

and the identity

$$\left(\partial_x + \frac{i}{\lambda}\right)(\exp(ipx + iqy)\,\Phi(x, y)) = \exp(ipx + iqy)\left(\partial_x - \frac{i}{\bar\lambda}\right)\Phi(x, y)\,, \tag{4.15}$$

where $\Phi(x, y)$ is an arbitrary function. Indeed, multiplying (4.12) by $\exp(-ipx - iqy)$, comparing the obtained equation with equation (4.6) with the change $\lambda \to -\bar\lambda$ and taking into account (4.14) and (4.15), one finds

$$N(x, y; \lambda\,\bar\lambda) = \exp(ipx + iqy)\,\chi(x, y; -\bar\lambda, -\lambda)\,. \tag{4.16}$$

So, we have the following $\bar\partial$-problem

$$\frac{\partial\chi(x, y; \lambda\,\bar\lambda)}{\partial\bar\lambda} = F(\lambda, \bar\lambda)\exp(ipx + iqy)\,\chi(x, y; -\bar\lambda, -\lambda)\,, \tag{4.17}$$

where the function $F(\lambda, \bar{\lambda})$ is given by (4.10). The $\bar{\partial}$-equation (4.17) generates the integral equation

$$\chi(x, y; \lambda, \bar{\lambda}) = 1 + \\ + \frac{1}{2\pi i} \iint_C \frac{d\mu \wedge d\bar{\mu}}{\mu - \lambda} F(\mu, \bar{\mu}) \exp(i\hat{p}x + i\hat{q}y)\, \chi(x, y; -\bar{\mu}, -\mu), \qquad (4.18)$$

where

$$\hat{p} = -\left(\frac{1}{\mu} + \frac{1}{\bar{\mu}}\right), \quad \hat{q} = i\left(\frac{1}{\bar{\mu}^2} - \frac{1}{\mu^2}\right)$$

via the generalized Cauchy formula.

Equation (4.18) is the inverse problem equation for the problem (4.3) and the function $F(\lambda, \bar{\lambda})$ is the inverse problem data. Equation (4.18) is uniquely solvable at least for small $F(\lambda, \bar{\lambda})$. Similar to the mKP-I case the solution χ of the inverse problem equations (4.18) obeys equation (4.3) for any constraint free inverse problem data $F(\lambda, \bar{\lambda})$.

The time dependence of the function $F(\lambda, \bar{\lambda})$ is defined similar to the mKP-I case. One has

$$F(\lambda, \bar{\lambda}; t) = F(\lambda, \bar{\lambda}; 0) \exp\left[-4i\left(\frac{1}{\lambda^3} + \frac{1}{\bar{\lambda}^3}\right)t\right] \qquad (4.19)$$

where $F(\lambda, \bar{\lambda}; 0)$ is an arbitrary function.

The formulae (4.19), (4.18) and (4.5) linearize the initial value problem for the mKP-II equation via the standard IST scheme.

For the real-valued «potential» $u(x, y, t)$ equation (4.6) with the use of (4.11) and (4.15) gives

$$\overline{\chi(x, y; \lambda, \bar{\lambda})} = \chi(x, y; -\bar{\lambda}, -\lambda). \qquad (4.20)$$

So, for the real-valued $u(x, y, t)$ the $\bar{\partial}$-problem (4.17) is of the form

$$\frac{\partial \chi(x, y; \lambda, \bar{\lambda})}{\partial \bar{\lambda}} = F(\lambda, \bar{\lambda}) \exp(ipx + iqy)\, \overline{\chi(x, y; \lambda, \bar{\lambda})}. \qquad (4.21)$$

In the terms of $\Psi(x, y, t)$ (4.2) one has

$$\frac{\partial \Psi(x, y; \lambda, \bar{\lambda})}{\partial \bar{\lambda}} = F(\lambda, \bar{\lambda})\, \overline{\Psi(x, y; \lambda, \bar{\lambda})}. \qquad (4.22)$$

Thus, for the real valued $u(x, y, t)$ the bounded solution of the

problem (4.1) is nothing but the pseudo (generalized)-analytical function in a sence of L. Bers [27] and I. Vekua [28]. The properties of the pseudo-analytic functions [27, 28] guarantee the solvability of equation (4.22) and hence, the inverse problem equations for the mKP-II equation with real $u(x, y, t)$ for any smooth data $F(\lambda, \bar{\lambda})$. Note that the similar situation takes place for the KP-II equation too (see e. g. [19]).

Emphasize that the inverse problem for (4.1) can be solved in a manner when one starts from the very beginning by the looking for its solutions within the class of pseudo (generalized)-analytic functions.

5. $\bar{\partial}$-DRESSING METHOD FOR THE mKP EQUATION

In the previous sections we solved the initial-value problem for the mKP equation and constructed exact solutions for the class of $u(x, y, t)$ which decrease as $O((x^2+y^2)^{-1})$ at $(x^2+y^2)^{1/2} \to \infty$. Much wider classes of exact solutions can be constructed by the $\bar{\partial}$-dressing method [20—22, 19].

The $\bar{\partial}$-dressing method is based on the use of the nonlocal $\bar{\partial}$-problem [20—22]

$$\frac{\partial \chi(\lambda, \bar{\lambda})}{\partial \bar{\lambda}} = (\chi * R)(\lambda, \bar{\lambda}) \overset{def}{=} \iint_C d\lambda' \wedge d\bar{\lambda}'\, \chi(\lambda', \bar{\lambda}')\, R(\lambda', \bar{\lambda}'; \lambda, \bar{\lambda}), \qquad (5.1)$$

where in our case χ and R are the scalar complex—valued functions. We assume that the function χ has the canonical normalization $(\chi \xrightarrow[\lambda \to \infty]{} 1)$ and the problem (5.1) is uniguelly solvable. The nonlocal $\bar{\partial}$-problem is a significant generalization of the nonlocal Riemann—Hilbert problem and quasi-local $\bar{\partial}$-problem which have appeared in sections 2 and 4.

A dependence on the variables x, y and t is introduced via the following dependence of R on x, y, t:

$$\frac{\partial R(\lambda', \bar{\lambda}'; \lambda, \bar{\lambda}; x, y, t)}{\partial x} = i\left(\frac{1}{\lambda'} - \frac{1}{\lambda}\right) R(\lambda', \bar{\lambda}'; \lambda, \bar{\lambda}; x, y, t),$$

$$\frac{\partial R(\lambda', \bar{\lambda}'; \lambda, \bar{\lambda}; x, y, t)}{\partial y} = \frac{1}{\sigma}\left(\frac{1}{\lambda'^2} - \frac{1}{\lambda^2}\right) R(\lambda', \bar{\lambda}'; \lambda, \bar{\lambda}; x, y, t), \qquad (5.2)$$

$$\frac{\partial R(\lambda',\bar{\lambda}';\lambda,\bar{\lambda};x,y,t)}{\partial t}=4i\left(\frac{1}{\lambda'^3}-\frac{1}{\lambda^3}\right)R(\lambda',\bar{\lambda}';\lambda,\bar{\lambda};x,y,t),$$

i.e.

$$R(\lambda',\bar{\lambda}';\lambda,\bar{\lambda};x,y,t)=\exp\left(i\frac{x}{\lambda'}+\frac{y}{\sigma\lambda'^2}+\frac{4it}{\lambda'^3}\right)\times$$

$$\times R_0(\lambda',\bar{\lambda}';\lambda,\bar{\lambda})\exp\left(-\frac{ix}{\lambda}-\frac{y}{\sigma\lambda^2}-\frac{4it}{\lambda^3}\right), \tag{5.3}$$

where $R_0(\lambda', \bar{\lambda}'; \lambda, \bar{\lambda})$ is an arbitrary function and $\sigma^2=\pm 1$. With the use of the «long» derivatives

$$D_x=\partial_x+\frac{i}{\lambda},\quad D_y=\partial_y+\frac{1}{\sigma\lambda^2},\quad D_t=\partial_t+\frac{4i}{\lambda^3} \tag{5.4}$$

equations (5.2) can be rewritten as

$$[D_x,R]=0,\quad [D_y,R]=0,\quad [D_t,R]=0. \tag{5.5}$$

A main problem consist in construction of operators L of the form

$$L=\sum_{n,l,m}u_{nlm}(x,y,t)\,D_x^n D_y^l D_t^m, \tag{5.6}$$

where $u_{nlm}(x, y, t)$ are some function, which obey the condition [20, 21]

$$\left[\frac{\partial}{\partial\bar{\lambda}},L\right]=0 \tag{5.7}$$

i.e. which have no singularities on λ. For such operator L the function $L\chi$ obeys the same $\bar{\partial}$-equation as the function χ. If there are several operators of this type then in virtue of the unigue solvability of equation (5.1), one has

$$L_i\chi=0. \tag{5.8}$$

The system (5.8) is just the linear system which generates the corresponding integrable equation [20—22].

In our case one can construct, as it is not difficult to show the two operators L_1 and L_2 which obey the condition (5.7), namely,

$$L_1=\sigma D_y+D_x^2+\sigma u D_x,$$

$$L_2=D_t+4D_x^3+6\sigma u D_x^2+\left(3\sigma u_x+\frac{3}{2}\sigma^2u^2-3\sigma^2\partial_x^{-1}u_y\right)D_x, \tag{5.9}$$

where

$$u(x,y,t)=-2\sigma^{-1}\frac{\partial}{\partial x}\ln\chi_0 \tag{5.10}$$

and $\chi_0\overset{def}{=}\chi(\lambda,\bar{\lambda};x,y,t)\,|_{\lambda=0}$ The corresponding linear system (5.8) is nothing but the system (1.2) with $V=\sigma u$, the compatibility of which is equivalent to the mKP equation.

Emphasize that the absence of the term of the form $p(x, y)\chi$ in (5.9) for any $R_0(\lambda', \bar{\lambda}'; \lambda, \bar{\lambda})$ is the consequence of the x, y, t dependence (5.3) of R via $1/\lambda$.

The formula (5.10) is, in fact, the dressing formula for the mKP equation. Indeed, starting with arbitrary given function $R_0(\lambda', \bar{\lambda}'; \lambda, \bar{\lambda})$ one firstly finds the solution χ of the $\bar{\partial}$-problem (5.1) which is equivalent to the integral equation

$$\chi(\lambda,\bar{\lambda};x,y,t)=1+\frac{1}{2\pi i}\iint\limits_C\frac{d\lambda'\wedge d\bar{\lambda}'}{\lambda'-\lambda}(\chi*R)(\lambda',\bar{\lambda}'), \tag{5.11}$$

where $R(\lambda', \bar{\lambda}'; \lambda, \bar{\lambda}; x, y, t)$ is given by (5.3) and then the formula (5.10) with

$$\chi_0=1+\frac{1}{2\pi i}\iint\limits_C\frac{d\lambda'\wedge d\bar{\lambda}'}{\lambda'}(\chi*R)(\lambda',\bar{\lambda}') \tag{5.12}$$

gives us the solution of the mKP equation.

Emphasize that within $\bar{\partial}$-dressing method nothing is assumed about the behaviour of $u(x, y, t)$ at $x, y\to\infty$.

Constraints on the data $R_0(\lambda', \bar{\lambda}'; \lambda, \bar{\lambda})$ which give rise to the decreasing at infinity solutions $u(x, y, t)$ of the mKP equation can be found by the consideration of small u similar to the KP equation [29]. In this case one has $\chi\sim 1$ and hence,

$$\chi_0=1+\Delta\overset{def}{=}1+\frac{1}{2\pi i}\iint\limits_C\frac{d\lambda'\wedge d\bar{\lambda}'}{\lambda'}\iint\limits_C\frac{d\lambda\wedge d\bar{\lambda}}{2\pi i}\times$$

$$\times R_0(\lambda,\bar{\lambda};\lambda',\bar{\lambda}')\exp\left[i\left(\frac{1}{\lambda'}-\frac{1}{\lambda}\right)x+\frac{1}{\sigma}\left(\frac{1}{\lambda'^2}-\frac{1}{\lambda^2}\right)y+4i\left(\frac{1}{\lambda'^3}-\frac{1}{\lambda^3}\right)t\right]. \tag{5.13}$$

From (5.10) and (5.13) it follows that the necessary conditions for decreasing of u at $x, y\to\infty$ are

$$\mathrm{Im}\left(\frac{1}{\lambda'}-\frac{1}{\lambda}\right)=0, \quad \mathrm{Re}\left(\frac{1}{\sigma}\left(\frac{1}{\lambda'^2}-\frac{1}{\lambda^2}\right)\right)=0. \tag{5.14}$$

At $\sigma=i$ the constraints (5.14) give $\mathrm{Im}\,\lambda'=\mathrm{Im}\,\lambda=0$ while at $\sigma=1$ one has $\lambda'=-\bar{\lambda}$. So, the decreasing solutions of the mKP equation correspond to the kernels R of the problem (5.1) of the form:
for mKP-I

$$R_0=T_{\mathrm{I}}(\lambda',\lambda)\,\delta(\lambda'-\bar{\lambda}')\,\delta(\lambda-\bar{\lambda}) \tag{5.15}$$

and for mKP-II

$$R_0=T_{\mathrm{II}}(\lambda,\bar{\lambda})\,\delta(\lambda'+\bar{\lambda}), \tag{5.16}$$

where T_{I} and T_{II} are arbitrary functions.

It is easy to see that for the kernels R of the form (5.15), (5.16) the nonlocal $\bar{\partial}$-problem (5.1) is reduced to the nonlocal Riemann—Hilbert problem (2.23) and the quasi-local $\bar{\partial}$-problem (4.16), respectively.

The consideration of the small u allows us also to find the constraints on R which guarantee the reality of $u(x, y, t)$. For the reality of u given by (5.10) it is sufficient that (for general u)

$$|\chi_0|=1 \tag{5.17}$$

at $\sigma=i$ and

$$\chi_0=\bar{\chi}_0 \tag{5.18}$$

at $\sigma=1$.

For small u the constraint (5.17) implies

$$\Delta=-\bar{\Delta}, \tag{5.19}$$

where Δ is defined in (5.13), while the constraint (5.18) gives

$$\Delta=\bar{\Delta}. \tag{5.20}$$

Changing the order of integration in (5.19) and (5.20), we obtain the following reality constraints:
for mKP-I

$$R_0(\lambda',\bar{\lambda}';\lambda,\bar{\lambda})\,\lambda'=\overline{R_0(\bar{\lambda},\lambda;\bar{\lambda}',\lambda')}\;\lambda \tag{5.21}$$

and for mKP-II

$$R_0(\lambda',\bar{\lambda}';\lambda,\bar{\lambda})=\overline{R_0(-\bar{\lambda}',-\lambda';-\bar{\lambda},-\lambda)}\;. \tag{5.22}$$

Iterating equation (5.11), it is not difficult to show that the conditions (5.21), (5.22) are the reality conditions for nonsmall $u(x, y, t)$ too.

In virtue of (5.3) the reality constraints (5.21) and (5.22) evidently are preserved in time.

6. EXACT SOLUTIONS VIA $\bar{\partial}$-DRESSING

The $\bar{\partial}$-dressing method allow us to construct several classes of exact solutions of the mKP equation.

6.1. Rational Solutions. Plane Lumps

We start with the construction of solutions with the rational dependence on x, y and t.

Let us choose the kernel R of the $\bar{\partial}$-problem in the form (see e.g [19])

$$R_0(\mu,\bar{\mu};\lambda,\bar{\lambda})=\frac{\pi}{2i}\cdot\sum_{k=1}^{N}S_k(\mu,\lambda)\,\delta(\lambda-\lambda_k)\,\delta(\mu-\lambda_k), \tag{6.1}$$

where $\delta(\lambda-\lambda_k)$ is the complex Dirac function, $S_k(\mu,\lambda)$ are some functions and $\lambda_1,\ldots,\lambda_N$ is the set of isolated points distinct from the origin.

For the kernel R_0 of the form (6.1), one has

$$\frac{\partial\chi(\lambda,\bar{\lambda})}{\partial\bar{\lambda}}=-\pi\sum_{k=1}^{N}\chi(\lambda_k)e^{F(\lambda_k)-F(\lambda)}S_k(\lambda_k,\lambda)\,\delta(\lambda-\lambda_k), \tag{6.2}$$

where

$$F(\lambda)\overset{def}{=}i\frac{x}{\lambda}+\frac{y}{\sigma\lambda^2}+\frac{4it}{\lambda^3}.$$

Then equation (5.11) gives at $\lambda\neq\lambda_k\,(k=1,\ldots,N)$

$$\chi(\lambda,\bar{\lambda})=1+\sum_{k=1}^{N}\frac{\chi(\lambda_k)S_k(\lambda_k,\lambda_k)}{\lambda-\lambda_k} \tag{6.3}$$

while in the limits $\lambda\to\lambda_i$, using (6.2), one gets

$$\chi(\lambda_i)=1-\frac{1}{2\pi i}\iint_C\frac{d\lambda'\wedge d\bar{\lambda}'}{\lambda'-\lambda_i}\times$$

$$\times \pi \sum_{k=1}^{N} \chi(\lambda_k)\, e^{F(\lambda_k)-F(\lambda')} S_k(\lambda_k, \lambda')\, \delta(\lambda'-\lambda_k)\,, \qquad i=1, \dots N. \tag{6.4}$$

The term in (6.4) witn $k=i$ is equal to

$$-\operatorname{Res} \frac{\chi(\lambda)\, e^{F(\lambda_i)-F(\lambda)} S_i(\lambda_i,\lambda)}{(\lambda-\lambda_i)^2}\Big|_{\lambda=\lambda_i} = \chi(\lambda_i)(S'(\lambda_i) - S_i(\lambda_i,\lambda_i) F'(\lambda_i))\,, \tag{6.5}$$

where

$$F'(\lambda_i) \overset{def}{=} \frac{\partial F(\lambda)}{\partial \lambda}\Big|_{\lambda=\lambda_i} = -\frac{ix}{\lambda_i^2} - \frac{2y}{\sigma\lambda_i^3} - \frac{12it}{\lambda_i^4}$$

and

$$S'(\lambda_i) \overset{def}{=} \frac{\partial S_i(\lambda_i,\lambda)}{\partial \lambda}\Big|_{\lambda=\lambda_i}.$$

As a result, equations (6.4) with different i give rise to system

$$\chi(\lambda_i)(1+S_i'(\lambda_i) - S_i F'(\lambda_i)) + \sum_{k\neq i} \frac{\chi(\lambda_k) S_k}{\lambda_k - \lambda_i} = 1\,, \qquad i=1, \dots, N. \tag{6.6}$$

Solving the system (6.6) with respect to $\chi(\lambda_i)$, one then finds the solution of the mKP equation by the formula (5.10) with

$$\chi_0 = 1 - \sum_{k=1}^{N} \frac{\chi(\lambda_k) S_k}{\lambda_k}. \tag{6.7}$$

Since $F'(\lambda_i)$ is the linear function on x, y, t then the solutions constructed are the rational functions on x, y and t. Note that the system of the type (6.6) which define the rational solutions has been derived for the first time in [20] within the dressing method based on the nonlocal Riemann—Hilbert problem.

The rational solutions constructed above are representable in the following compact form

$$u = -2\sigma^{-1} \frac{\partial}{\partial x} \ln \det(1 + B\cdot A^{-1})\,, \tag{6.8}$$

where the $N\times N$ matrices A and B are defined by

$$A_{ik} = \delta_{ik}(1+S_i' - S_i F'(\lambda_i)) - (1-\delta_{ik}) \frac{S_k}{\lambda_i - \lambda_k}, \tag{6.9}$$

$$B_{ik} = -S_k \cdot \lambda_k^{-1}. \tag{6.10}$$

The reality condition of u implies certain constraints on $\lambda_1, \dots, \lambda_N$ and $S_1, \dots, S_N$. We will consider these constraints for the particular choice of S_i, namely, $S_i = -i\lambda_i^2$ which is motivated by the form of the lump solutions (3.9), (3.7) of the mKP-I equation. In this case

$$A_{ik} = \delta_{ik}\left(x + i\frac{2y}{\sigma\lambda_i} + \frac{12t}{\lambda_i^2} + \gamma_i\right) + (1-\delta_{ik}) \frac{i\lambda_k^2}{\lambda_i - \lambda_k}, \qquad B_{ik} = i\lambda_k. \tag{6.11}$$

where we denote $\gamma_i \overset{def}{=} 1 + S_i'$.

For the mKP-I equation $(\sigma = i)$ the solutions (6.8) are real and bounded in the two cases.

The first case: $N=2n$ and

$$\lambda_{n+i} = \bar{\lambda}_i, \quad \gamma_i = -\frac{i\lambda_i}{2} + c_i, \quad \gamma_{n+i} = -\frac{i\bar{\lambda}_i}{2} + \bar{c}_i \quad (i=1,\dots,n). \tag{6.12}$$

where λ_i $(i=1, \dots, n)$ are arbitrary isolated points outside the real axis and c_i are arbitrary constants. In this case the solutions (6.8) are nothing but the multilump solutions (3.9) of the mKP-I equation found in section 3.

The second case: arbitrary N and

$$\operatorname{Im}\lambda_i = 0, \quad \gamma_i = -\frac{i\lambda_i}{2} + c_i, \quad \operatorname{Im} c_i = 0. \tag{6.13}$$

The corresponding solutions (6.8) are bounded but they do not decay in some directions. The simplest solution of this type is $(N=1)$

$$u(x,y,t) = \frac{2\lambda_1}{\left(x - \frac{2y}{\lambda_1} + \frac{12t}{\lambda_1^2} + \delta_1\right)^2 + \frac{\lambda_1^2}{4}}. \tag{6.14}$$

This solution does not decrease at the direction $x - (2y/\lambda_1) = \text{const}$, i.e. it is the plane lump of the mKP-I equation.

The next solution of this type is of the form $(N=2)$

$$u(x,y,t) = -\frac{2\lambda_1 X_2^2 + 2\lambda_2 X_1^2 + \frac{\lambda_1\lambda_2}{2}\left(\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}\right)^2}{\left[X_1X_2 - \frac{\lambda_1\lambda_2}{4}\left(\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}\right)^2\right]^2 + \frac{1}{4}(\lambda_1 X_2 + \lambda_2 X_1)^2}, \tag{6.15}$$

where

$$X_i = x - \frac{2y}{\lambda_i} + \frac{12t}{\lambda_i^2} + \delta_i, (i=1,2)$$

and δ_i are arbitrary real constants. It is easy to see that the solution (6.15) describes the scattering of the two lumps of the form (6.14). This solution is nonsingular at $\lambda_1 + \lambda_2 \neq 0$.

General solution (6.8) in the case (6.13) is bounded if all $\lambda_i > 0$ ($i=1, ..., n$), does not decrease at the directions $x - (2y/\lambda_i) = \text{const}$ and describes the scattering of N plane lumps (6.14). It is easy to see that the scattering of the plane lumps is completely trivial: the phase shift is absent.

Emphasize that the plane lumps have not been found for the KP, DS and Ishimori equations. So their existence for the mKP-I equation is the novelty for the 2+1-dimensional soliton equations.

In addition to the pure cases (6.12) and (6.13) one can consider also the general mixed case in which one has $2n$ points of the type (6.12) and N points of the type (6.13). Such solutions of the mKP-I equation are given by the formula (6.8), (6.11) where A is the $(2n+N)\times(2n+N)$ matrix of the form (6.11) with $(\lambda_1, ..., \lambda_{2n+N}) \equiv (\lambda_1, ..., \lambda_n, \bar{\lambda}_1, ..., \bar{\lambda}_n; \alpha_1, ..., \alpha_N)$, $(\text{Im}\,\alpha_k = 0)$. They describe the scattering of n decaying lumps (3.10) and N plane lumps (6.14).

For the mKP-II equation the situation is completely different. Similar to the mKP-I case the real-valued u arises into the two cases:

1) $N=2n$; $\lambda_{k+n} = \bar{\lambda}_k$, $\gamma_{k+n} = \bar{\gamma}_k$, $(k=1, ..., n)$;

2) arbitrary N, $\lambda_k = i\alpha_k$ $(\text{Im}\,\alpha_k = 0)$; $\gamma_k = \bar{\gamma}_k$ $(k=1, ..., N)$. (6.16)

But, as it not difficult to see, all these rational solutions of the mKP-II equation are singular. For instance, the analog of solution (6.14) looks like

$$u(x,y,t) = \frac{2\alpha_1}{\frac{\alpha_1^2}{4} - \left(x + \frac{2y}{\alpha_1} - \frac{12t}{\alpha_1^2}\right)^2}. \tag{6.17}$$

This solution describes the uniform motion of the two simple pole, plane, opposite sign singularities (located alone the lines $x + (2y/\alpha_1) = \text{const}$) which parallel to each other (with the distance α_1) and move with the same velocity $12\alpha_1^{-2}$. The mKP equation has

also rational solutions which correspond to the multiple pole singularities of the eigenfunction χ. They will be considered elsewhere.

6.2. Solutions with Functional Parameters

Another general class of exact solutions corresponds to the degenerated kernel $R_0(\lambda', \bar{\lambda}'; \lambda, \bar{\lambda})$, i.e.

$$R_0(\lambda', \bar{\lambda}'; \lambda, \bar{\lambda}) = \pi \sum_{k=1}^{n} f_k(\lambda', \bar{\lambda}')\, g_k(\lambda, \bar{\lambda}). \tag{6.18}$$

For the kernel R_0 of this type equation (5.11) gives

$$\chi(\lambda, \bar{\lambda}) = 1 + \pi \sum_{k=1}^{n} h_k(x) \iint_C \frac{d\lambda' \wedge d\bar{\lambda}'}{2\pi i(\lambda' - \lambda)} g_k(\lambda', \bar{\lambda}')\, e^{-F(\lambda')}, \tag{6.19}$$

where

$$h_k(x) \stackrel{\text{def}}{=} \iint_C d\lambda \wedge d\bar{\lambda}\, \chi(\lambda, \bar{\lambda})\, e^{F(\lambda)} f_k(\lambda, \bar{\lambda}) \tag{6.20}$$

and $F(\lambda) = \frac{ix}{\lambda} + \frac{y}{\sigma\lambda^2} + \frac{4it}{\lambda^3}$. The quantities h_k are calculated from the algebraic system

$$\sum_{k=1}^{n} A_{lk} h_k = \xi_l \quad (l=1, ..., n), \tag{6.21}$$

where

$$\xi_l(x,y,t) \stackrel{\text{def}}{=} \iint_C d\lambda \wedge d\bar{\lambda}\, e^{F(\lambda)} f_l(\lambda, \bar{\lambda}) \tag{6.22}$$

and

$$A_{lk} \stackrel{\text{def}}{=} \delta_{lk} + \pi \iint_C d\lambda \wedge d\bar{\lambda} \iint_C \frac{d\lambda' \wedge d\bar{\lambda}'}{2\pi i} \frac{e^{F(\lambda) - F(\lambda')} f_l(\lambda, \bar{\lambda})\, g_k(\lambda', \bar{\lambda}')}{\lambda - \lambda'}. \tag{6.23}$$

The system (6.21) arises from (6.19) after multiplication by $e^{F(\lambda)} f_l(\lambda, \bar{\lambda})$ and integration over λ.

Solving the system (6.21) for arbitrary given functions f_l and g_l, one finds the solution of the mKP equation by the formula (5.10) with

$$\chi_0 = 1 + \sum_{k=1}^{n} h_k(x)\, \eta_k(x) = 1 + \frac{1}{2i} \sum_{k,l} \eta_k (A^{-1})_{kl} \xi_l, \tag{6.24}$$

where

$$\eta_k(x) = \iint_C \frac{d\lambda \wedge d\bar{\lambda}}{\lambda} g_k(\lambda, \bar{\lambda})\, e^{-F(\lambda)}. \tag{6.25}$$

The compact formula for such solutions is of the form

$$u = -2\sigma^{-1} \frac{\partial}{\partial x} \ln \det(1 + B \cdot A^{-1}), \tag{6.26}$$

where A is given by (6.23) and $B_{kl} = (1/2i)\xi_k \eta_l$.

The solutions (6.26) are parametrized by the $2n$ arbitrary complex-valued functions f_k, g_k $(k=1, \ldots, n)$.

Similar to the section 3 the matrices A and $A+B$ can be rewritten in the term of ξ_l and η_l only. Namely, one has

$$u = -2\sigma^{-1} \frac{\partial}{\partial x} \ln \det(\tilde{A} \cdot A^{-1}), \tag{6.27}$$

where

$$A_{kl} = \delta_{kl} - \frac{1}{2i} \partial_x^{-1}(\eta_l \xi_{kx}), \quad \tilde{A}_{kl} = \delta_{kl} + \frac{1}{2i} \partial_x^{-1}(\eta_{lx} \xi_k), \tag{6.28}$$

where η_k, ξ_l are arbitrary functions of the form (6.25) and (6.22). The integration ∂_x^{-1} tn (6.28) is choosen in a way which quarantees the existence of A and $\tilde{A}$.

The functions ξ_l and η_k are the solutions of the linearized mKP equation. So, the formula (6.27) presents the class of solutions of the mKP equation which is parametrized by $2n$ arbitrary complex valued solution (not necessarily decreasing) of the linearized mKP equation.

The reality conditions (5.21) and (5.22) imply certain constraints on the functions f_k and g_k $(k=1, \ldots, n)$. They are satisfied, in particular, if

$$R_k g_k(\lambda, \bar{\lambda}) = \overline{f_k(\bar{\lambda}, \lambda)}\ \lambda \tag{6.29}$$

for the mKP-I equation where R_k are arbitrary real constants and

$$\overline{f_k(\lambda, \bar{\lambda})} = f_k(-\bar{\lambda}, -\lambda), \quad \overline{g_k(\lambda, \bar{\lambda})} = g_k(-\bar{\lambda}, -\lambda), \tag{6.30}$$

for the mKP-II equation. The condition (6.29) implies

$$\bar{\xi}_k = -R_k \eta_k \tag{6.31}$$

and, hence, as one can show from (6.27), the real-valued solutions (6.27) of the mKP-I equation can be represented as

$$u = 4 \frac{\partial}{\partial x} \arg \det C, \tag{6.32}$$

where $C_{kl} = A_{kl} R_l = R_k \delta_{kl} + (1/2i)\partial_x^{-1}(\xi_{kx} \bar{\xi}_l)$. These solutions generalize the decaying solutions (3.27), (3.29).

Correspondingly, the condition (6.30) for the mKP-II equation means

$$\bar{\xi}_k = -\xi_k, \quad \bar{\eta}_k = \eta_k. \tag{6.33}$$

6.3. Plane Soliton and Breathers

The class of exact solutions with functional parameters constructed in section (6.2) contains as the particular cases the plane solitons and breathers of the mKP equation.

The real-valued plane solitons of the mKP-I equation correspond to the choice

$$f_k(\lambda, \bar{\lambda}) = R_k \delta(\lambda - \lambda_k), \quad g_k(\lambda, \bar{\lambda}) = \lambda \delta(\lambda - \bar{\lambda}_k), \tag{6.34}$$

where R_k are arbitrary real constants, i.e.

$$\xi_l(x, y, t) = -2iR_l \exp(F(\lambda_l)), \quad \eta_k(x, y, t) = -2i \exp(-F(\bar{\lambda}_k)). \tag{6.35}$$

In this case the solutions are of the form

$$u(x, y, t) = 4 \frac{\partial}{\partial x} \arg \det A, \tag{6.36}$$

where

$$A_{lk} = \delta_{lk} + 2i \frac{R_k \bar{\lambda}_k}{\lambda_l - \bar{\lambda}_k} \exp(F(\lambda_k) - F(\bar{\lambda}_k)) \tag{6.37}$$

and

$$F(\lambda_k) \overset{def}{=} i\left(\frac{x}{\lambda_k} - \frac{y}{\lambda_k^2} + \frac{4t}{\lambda_k^3}\right).$$

The simplest soliton looks like $(n=1,\ \lambda_1 = \lambda_R + i\lambda_I)$

$$u(x, y, t) = -4 \frac{2 \frac{\lambda_I}{|\lambda|^2} \operatorname{Sgn} R_1}{e^{2f} + \left(e^{-f} + \frac{\lambda_R}{\lambda_I} (\operatorname{Sgn} R_1)\, e^{f}\right)^2}, \tag{6.39}$$

where

$$f=\frac{\lambda_I}{|\lambda|^2}\left(x-\frac{2\lambda_R}{|\lambda|^2}y-\frac{4(\lambda_I^2-3\lambda_R^2)t}{|\lambda|^4}+\frac{|\lambda|^2}{\lambda_I}\ln|R_1|\right). \qquad (6.40)$$

This solution evidently regular in x and y and is a constant alone the direction $x-\frac{2\lambda_R}{|\lambda|^2}y=\text{const}$, i.e. it is the plane soliton. General solution (6.31) describes the scattering of n plane solitons of the form (6.33). At $\lambda_R=0$ the solution (6.33) is reduced to the mKdV soliton (see e. g. [2]).

The plane soliton's eigenfunctions in the points λ_k can be found from the system

$$\chi(\lambda_k)+2i\sum_{l=1}^{n}\frac{R_l\bar{\lambda}_l}{\lambda_k-\bar{\lambda}_l}e^{F(\lambda_l)-F(\bar{\lambda}_l)}\chi(\lambda_l)=1\,, \qquad k=1,\dots,n \qquad (6.41)$$

which follows from (6.19).

For the mKP-II equation the real plane solitons in virtue of (6.30), correspond to the kernel R_0 of the form

$$R_0=\sum_k R_k\,\delta(\lambda-i\alpha_k)\,\delta(\mu-i\beta_k)\,, \qquad (6.42)$$

where R_k, α_k and β_k are arbitrary real constants, i.e.

$$\xi_l=-2iR_l\exp(F(i\alpha_l))\,, \qquad \eta_l=-2\beta_l^{-1}\exp(-F(i\beta_l)) \qquad (6.43)$$

in (6.27) and (6.28). These solutions of the mKP-II equation are

$$u(x,y,t)=-2\frac{\partial}{\partial x}\ln\det(1+B\cdot A^{-1})\,, \qquad (6.44)$$

where

$$A_{km}=\delta_{km}+\frac{R_m}{\alpha_k-\beta_m}\exp(F(i\alpha_m)-F(i\beta_m)) \qquad (6.45)$$

and

$$B_{km}=2R_m\beta_m^{-1}\exp(F(i\alpha_m)-F(i\beta_m))\,. \qquad (6.46)$$

The simplest plane soliton of the mKP-II equation is $(n=1)$

$$u(x,y,t)=-\frac{2(\alpha-\beta)^2}{\alpha\beta^2}\,\frac{\varepsilon}{\left(e^{-f}-\frac{\alpha}{\beta}\varepsilon e^{f}\right)(e^{-f}-\varepsilon e^{f})}\,, \qquad (6.47)$$

where

$$2f=\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)x-\left(\frac{1}{\alpha^2}-\frac{1}{\beta^2}\right)y-4\left(\frac{1}{\alpha^3}-\frac{1}{\beta^3}\right)t+\ln 2\left|\frac{R}{\beta-\alpha}\right|$$

and $\varepsilon=\mathrm{Sgn}(R/(\beta-\alpha))$. The solution (6.47) is nonsingular at $\varepsilon<0$ and $\alpha\beta^{-1}>0$. Note that the general formula for the mKP-II plane solitons different from (6.44) has been derived in [24] within the τ-function approach.

The real-valued solutions of the mKP equation of the breather type correspond to the kernel R_0 with the even number of delta-function contributions. For the mKP-1 equation in virtue of (5.21), the suitable kernel R_0 is

$$R_0(\lambda',\bar{\lambda}';\lambda,\bar{\lambda};0)=$$

$$=\pi\lambda\sum_{k=1}^{n}\left[R_k\delta(\lambda'-\lambda_k^+)\,\delta(\lambda-\lambda_k^-)+\bar{R}_k\,\delta(\lambda'-\overline{\lambda_k^-})\,\delta(\lambda-\overline{\lambda_k^+})\right], \qquad (6.48)$$

where R_k, λ_k^+ and λ_k^- $(k=1,\dots,n)$ are arbitrary complex parameters. The corresponding real solutions of the mKP-1 equation are of the form

$$u(x,y,t)=2i\frac{\partial}{\partial x}\ln\frac{\overline{\det A}}{\det A}=4\frac{\partial}{\partial x}\arg\det A\,, \qquad (6.49)$$

where A is the $2n\times 2n$ matrix:

$$A_{km}=\delta_{km}+2i\frac{\tilde{R}_m\tilde{\lambda}_m}{\lambda_k-\tilde{\lambda}_m}\exp(F(\lambda_m)-F(\tilde{\lambda}_m)) \quad (k,m=1,\dots,2n) \qquad (6.50)$$

where

$$(\lambda_1,\dots,\lambda_{2n})\stackrel{def}{=}(\lambda_1^+,\dots,\lambda_n^+;\overline{\lambda_1^-},\dots,\overline{\lambda_n^-})\,,$$
$$(\tilde{\lambda}_1,\dots,\tilde{\lambda}_{2n})\stackrel{def}{=}(\lambda_1^-,\dots,\lambda_n^-;\overline{\lambda_1^+},\dots,\overline{\lambda_n^+})\,, \qquad (6.51)$$
$$(\tilde{R}_1,\dots,\tilde{R}_{2n})\stackrel{def}{=}(R_1,\dots,R_n;\bar{R}_1,\dots,\bar{R}_n)\,.$$

The solutions (6.49) are rather complicated. But it is not difficult to see that they are of the breather type. For instance, in the simplest case $n=1$ the $\det A$ looks like

$$\det A=1+e^{f}\sum_{k=1}^{2}a_k e^{i(\varphi+\delta_k)}+be^{2f}, \qquad (6.52)$$

where a_1, a_2, b are certain constants and

$$f=\left(\frac{\lambda_{1I}}{|\lambda_1|^2}-\frac{\lambda_{2I}}{|\lambda_2|^2}\right)x+4\left(\frac{\lambda_{2R}\lambda_{2I}}{|\lambda_2|^4}-\frac{\lambda_{1R}\lambda_{1I}}{|\lambda_1|^4}\right)y+$$

$$+4\left(\frac{\lambda_{1I}^3+3\lambda_{1R}^2\lambda_{1I}}{|\lambda_1|^6}-\frac{\lambda_{2I}^3+3\lambda_{2R}^2\lambda_{2I}}{|\lambda_2|^6}\right)t, \tag{6.53}$$

$$\varphi=\left(\frac{\lambda_{1R}}{|\lambda_1|^2}-\frac{\lambda_{2R}}{|\lambda_2|^2}\right)x-2\left(\frac{\lambda_{1R}^2-\lambda_{1I}^2}{|\lambda_1|^4}-\frac{\lambda_{2R}^2-\lambda_{2I}^2}{|\lambda_2|^4}\right)y+$$

$$+4\left(\frac{\lambda_{1R}^3-3\lambda_{1R}\lambda_{1I}^2}{|\lambda_1|^6}-\frac{\lambda_{2R}^3-3\lambda_{2R}\lambda_{2I}^2}{|\lambda_2|^6}\right)t.$$

So, the solution (6.51) contains both oscillations and decreasing terms. This simplest solution at $\lambda_1^+=-\bar{\lambda}_1^-$ is reduced to the well-known breather of the mKdV equation (see e. g. [2]).

6.4. Other Exact Solutions

Here we present the two particular cases of the solutions (6.43) of the mKP-I equation. The first one corresponds to $\lambda_1^+=i\nu_1$, $\lambda^-=i\nu_2$ where ν_1 and ν_2 are arbitrary real constants and $R_1=\bar{R}_2=b$. It is of the form

$$u(x,y,t)=8\left(\frac{1}{\nu_2}-\frac{1}{\nu_1}\right)\frac{e^{-f}\cos\varphi\left(1-\left(\frac{\nu_1+\nu_2}{\nu_1-\nu_2}\right)^2e^{2f}\right)}{\left(e^{-f}-\frac{2(\nu_1+\nu_2)}{\nu_1-\nu_2}\sin\varphi+\left(\frac{\nu_1+\nu_2}{\nu_1-\nu_2}\right)^2e^f\right)^2+4\cos^2\varphi}, \tag{6.54}$$

where

$$f(x,t)=\left(\frac{1}{\nu_1}-\frac{1}{\nu_2}\right)x-4\left(\frac{1}{\nu_1^3}-\frac{1}{\nu_2^3}\right)t+\ln|b|,$$

$$\varphi(y)=\left(\frac{1}{\nu_1^2}-\frac{1}{\nu_2^2}\right)y+\delta, \quad b=|b|\,e^{i\delta}. \tag{6.55}$$

The solution (6.54) is periodic in y and has a soliton behaviour along the coordinate x.

Another special breather type solution corresponds to $\lambda^+=-\overline{\lambda^-}=\lambda$, $R_1=\bar{R}_2=b$ and looks like

$$u(x,y,t)=\frac{4\lambda_R e^f\sin\varphi\left(1-\frac{|\lambda|^4}{\lambda_R^2\lambda_I^2}e^{2f}\right)}{\left(1-\frac{2\lambda_I}{\lambda_R}e^f\cos\varphi-\frac{(\lambda_R^2-\lambda_I^2)}{\lambda_R^2\lambda_I^2}|\lambda|^2e^{2f}\right)^2+4\left(\frac{|\lambda|^2}{\lambda_R\lambda_I}e^{2f}-e^f\cos\varphi\right)^2}, \tag{6.56}$$

where

$$f(y)=-4\frac{\lambda_R\lambda_I}{|\lambda|^4}y+\ln|b|,$$

$$\varphi(x,t)=\frac{2\lambda_R}{|\lambda|^2}x+\frac{8(\lambda_R^3-3\lambda_R\lambda_I^2)t}{|\lambda|^6}+\delta, \quad b=|b|\,e^{i\delta}. \tag{6.57}$$

This solution decreases exponentially at $|y|\to\infty$. It is periodic in x and move along the axis x with the velocity $4\frac{3\lambda_I^2-\lambda_R^2}{|\lambda|^4}$.

In a similar manner one can construct the breathers for the mKP-II equation. So, the structure of the breather type solutions of the mKP equation is rather rich.

Note that the solutions (6.54) and (6.56) are not bounded on the whole plane (x, y). The singularity of the solution (6.56), as it is not difficult to see, is of the type $\varepsilon^{-1/2}$ $(\varepsilon\to 0)$ at the discrete points, i.e. it is integrable $(u\in L_1)$.

In addition to the solutions of the mKP equation enumerated above one can construct also the general solutions of the mixed type which «contains» the solutions of the different type (lumps, plane solitons etc.). For instance, the solutions of the mKP-I equation which contains both plane lumps and plane solitons correspond to the kernel

$$R_0(\lambda,\bar{\lambda};\mu,\bar{\mu})=$$

$$=\pi\mu\sum_{k=1}^{n_1}S_k(\lambda,\mu)\delta(\lambda-\alpha_k)\,\delta(\mu-\alpha_k)+\pi\mu\sum_{l=1}^{n_2}R_l\,\delta(\lambda-\lambda_l)\delta(\mu-\bar{\lambda}_l), \tag{6.58}$$

where α_k $(k=1,\ldots,n_1)$ are real constants, R_l are real constants and $S_k(\lambda,\mu)=\overline{S_k(\bar{\mu},\bar{\lambda})}$.

These solutions are of the form

$$u(x,y,t)=4\frac{\partial}{\partial x}\arg\det A, \tag{6.59}$$

where A is the $(n_1+n_2)\times(n_1+n_2)$ matrix with elements

$$A=\begin{pmatrix}A_{kj}, & A_{kN}\\ A_{Mj}, & A_{MN}\end{pmatrix}, \tag{6.60}$$

where

$$A_{kj}=\delta_{kj}\left(x-\frac{2y}{\alpha_k}+\frac{12t}{\alpha_k^2}-\frac{i\alpha_k}{2}+\delta_k\right)+i(1-\delta_{kj})\frac{\alpha_j^2}{\alpha_k-\alpha_j}, \quad k,j=1,\ldots,n_1;$$

$$A_{kN}=\frac{2iR_N\bar{\lambda}_N}{\alpha_k-\bar{\lambda}_N}\exp\left(F(\lambda_N)-F(\bar{\lambda}_N)\right), \quad k=1,\ldots,n_1, \quad N=1,\ldots,n_2; \tag{6.61}$$

$$A_{Mj}=\frac{i\alpha_j^2}{\lambda_M-\alpha_j},\quad M=1,\dots,n_2,\quad j=1,\dots,n_1;$$

$$A_{MN}=\delta_{NM}+\frac{2iR_N\bar{\lambda}_N}{\lambda_M-\bar{\lambda}_N}\exp(F(\lambda_N)-F(\bar{\lambda}_N)),\quad M,N=1,\dots,n_2.$$

The simplest solution of this type describes the scattering of the plane lump (6.17) and plane soliton (6.33):

$$u(x,y,t)=\frac{2\alpha+4RDe^{2g}}{A^2+B^2},$$

where

$$D=\frac{\alpha\lambda_R}{\lambda_I}+\frac{\alpha R}{2}\frac{|\lambda|^2}{\lambda_I^2}e^{2g}+\frac{4\alpha^2}{|\lambda|^2}\frac{\lambda_I^2}{|\lambda-\alpha|^2}X_\alpha-\frac{2\lambda_I}{|\lambda|^2}\left(X_\alpha^2+\frac{\alpha^2}{4}\right),$$

$$A=\left(X_\alpha+\frac{\alpha b}{2}\right)Re^{2g}+\frac{\alpha}{2},\quad B=X_\alpha\left(1+\frac{R\lambda_R}{\lambda_I}e^{2g}\right)-\frac{\alpha R}{2}ae^{2g},$$

$$a=\left|\frac{\lambda+\alpha}{\lambda-\alpha}\right|^2,\quad b=\frac{\lambda_R}{\lambda_I}-\frac{4\alpha\lambda_I}{|\lambda-\alpha|^2},$$

$$g(x,y,t)=\frac{x\lambda_I}{|\lambda|^2}-\frac{2y\lambda_R\lambda_I}{|\lambda|^4}+\frac{4t(3\lambda_R^2\lambda_I-\lambda_I^3)}{|\lambda|^6},$$

$$X_\alpha=x-\frac{2y}{\alpha}+\frac{12t}{\alpha^2}+\delta.$$

7. THE MIURA TRANSFORMATION BETWEEN THE mKP AND KP EQUATIONS

The Miura transformation

$$u_{KP}=-\frac{1}{2}\sigma\partial_x^{-1}V_y-\frac{1}{2}V_x-\frac{1}{4}V^2, \tag{7.1}$$

which maps the solutions V of the mKP equation (1.1) into the solutions u_{KP} of the KP equation

$$u_{KPt}+u_{KPxxx}+6u_{KP}u_{KPx}+3\sigma^2\partial_x^{-1}u_{KPyy}=0 \tag{7.2}$$

has been found in [23] within the framework of the gauge invariant description of the KP equation and in [24] in the τ-function method. The 2+1-dimensional Miura transformation (7.1) is similar, in general, to the well-known Miura transformation between the mKdV

and KdV equations but in some respects, as we will show, there are the essential differences between them.

Similar to the 1+1-dimensional case the Miura transformation (7.1) deeply interrelates the algebraic structures associated with the mKP and KP equations (see e. g. [30]).

As for as concerning the interrelation between the classes of solutions the situation is quite different and more interesting in the 2+1-dimensional case. Indeed, the real-valued solutions $U_{mKP\text{-}I}$ of the mKP-I equation give rise to the following solutions of the KP-I equation

$$u_{KP\text{-}I}=\frac{1}{2}\partial_x^{-1}u_{mKP\text{-}I\,y}-\frac{i}{2}u_{mKP\text{-}I\,x}+\frac{1}{4}u^2_{mKP\text{-}I}. \tag{7.3}$$

So, the Miura transformation even does not convert the real solutions of the mKP-I equation into the real solutions of the KP-I equation.

On the other hand, for the mKP-II case one has

$$u_{kp\text{-}II}=-\frac{1}{2}\partial_x^{-1}u_{mKP\text{-}II\,y}-\frac{1}{2}u_{mKP\text{-}II\,x}-\frac{1}{4}u^2_{mKP\text{-}II}. \tag{7.4}$$

So the Miura transformation interrelates the real solutions of the mKP-II and KP-II equations. Moreover, since $u_{mKP\text{-}II}=-2(\ln\chi_0)_x$ where χ_0 is the mKP-II eigenfunction at $\lambda=0$, one gets

$$u_{KP\text{-}II}(x,y,t)=-\frac{(\chi_0^{-1})_y+(\chi_0^{-1})_{xx}}{\chi_0^{-1}} \tag{7.5}$$

or

$$(\chi_0^{-1})_y+(\chi_0^{-1})_{xx}+u_{KP\text{-}II}\chi_0^{-1}=0. \tag{7.6}$$

Equation (7.6) evidently indicates that $u_{KP\text{-}II}$ is the solution of the KP-II equation while the formula (7.5) allows us to construct the solutions of the KP-II equation using χ_0 found for mKP-II equation.

Let us consider at first the simplest one soliton solution (6.47) of the mKP-II equation. For this solution

$$\chi_0=\frac{1-\frac{\alpha}{\beta}\varepsilon e^{f}}{1-\varepsilon e^{f}}. \tag{7.7}$$

Substituting this expression into (7.5), one obtains

$$u_{\text{KP-II}} = -2\varepsilon \frac{(\alpha-\beta)^2}{\alpha\beta^3} \frac{1}{\left(e^{-\tilde f/2} - \frac{\alpha}{\beta}\varepsilon e^{\tilde f/2}\right)^2} =$$

$$= +2^{-1}\left(\frac{1}{\alpha} - \frac{1}{\beta}\right)^2 \operatorname{ch}^{-2}(\tilde f/2)\,, \tag{7.8}$$

where

$$\tilde f(x,y,t) = x\left(\frac{1}{\alpha} - \frac{1}{\beta}\right) - y\left(\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right) - 4t\left(\frac{1}{\alpha^3} - \frac{1}{\beta^3}\right) + \ln\frac{2R\alpha}{\beta(\alpha-\beta)}.$$

The solution (7.8) is nothing but the well-known plane soliton of the mKP-II equation (see, e. g. [1]). The function (7.8) is nonsingular one not only for the same values of parameters α, β, ε ($\varepsilon<0$, $\alpha/\beta>0$) as the mKP-II plane soliton (6.47) but also for $\varepsilon>0$, $\alpha/\beta<0$ for which the soliton (6.47) is the singular one. The properties of the KP-II plane soliton (7.8) are quite different in these two cases. Namely, the solution (7.8) at $\varepsilon>0$, $\alpha/\beta<0$ (type I) is the standard plane soliton of the KP-II equation which is reduced at $\alpha=-\beta$ to the $1+1$-dimensional KdV soliton

$$u_{\text{KdV}} = +\frac{2}{\alpha^2}\,\frac{1}{\operatorname{ch}^2(\varphi/2)}\,, \quad \varphi = \tilde f\,|_{\alpha=-\beta} \tag{7.9}$$

while for $\varepsilon<0$, $\alpha/\beta>0$ (type II) the solution (7.8) does not admit the nontrivial one-dimensional limit ($u_{(7.8)}/_{\alpha=\beta}=0$).

So we see that the Miura transformation (7.4) maps, the plane soliton (6.47) of the mKP-II equation into the plane soliton (7.8) of the KP-II equation. More precisely, it maps the bounded plane soliton of the mKP-II equation into the type II plane soliton of the KP-II equation and the singular plane soliton of the mKP-II into the standard (type I) plane soliton of the KP-II equation.

Similar situation takes place for the general plane soliton solutions. This can be proved directly by the substitution of the general expression for χ_0 given by (6.24), (6.45), (6.46) into (7.5). But it is more convenient to use the relation

$$\chi_{0y} + \chi_{0xxx} - 2\frac{\chi_{0x}^2}{\chi_0} + 2i\chi_{1x} - 2i\frac{\chi_{0x}}{\chi_0}\chi_1 = 0\,, \tag{7.10}$$

where $\chi(\lambda) = \chi_0 + \lambda\chi_1 + \ldots$ which follows directly from (4.3). Taking into account (7.10), one gets

$$u_{\text{KP-II}} = -2i\frac{\partial}{\partial x}\left(\frac{\chi_1}{\chi_0}\right). \tag{7.11}$$

Then for the plane soliton solutions of the mKP-II equation, we have

$$\chi_0 = \det(1 + BA^{-1})\,, \quad \chi_1 = \operatorname{tr}(B_1 A^{-1})\,, \tag{7.12}$$

where the matrices A and B are given by (6.45), (6.46) and from (6.19), (6.43) one can obtain the following expression for matrix B_1:

$$(B_1)_{lk} = \frac{1}{2}\xi_l \eta_{kx} = 2iR_l\beta_k^{-2}\exp\left(F(i\alpha_l) - F(i\beta_k)\right). \tag{7.13}$$

Substituting (7.12). (7.13) into (7.11), after some transformations, one gets

$$u_{\text{KP-II}} = 2\frac{\partial^2}{\partial x^2}\ln\det A_{\text{KP}}\,, \tag{7.14}$$

where

$$(A_{\text{KP}})_{nm} = \delta_{nm} - \frac{2\beta_n^{-2}R_n\exp\left[x\left(\frac{1}{\alpha_n} - \frac{1}{\beta_m}\right) - y\left(\frac{1}{\alpha_n^2} - \frac{1}{\beta_m^2}\right) - 4t\left(\frac{1}{\alpha_n^3} - \frac{1}{\beta_m^3}\right)\right]}{\alpha_n^{-1} - \beta_m^{-1}}$$

that is the known general formula for the plane multisolitons of the KP-II equation (see e. g. [1]). It represents the two types of plane solitons. The type I admits the $1+1$-dimensional reduction while the solutions of the type-II are pure $2+1$-dimensional one.

Thus, the Miura transformation (7.4) maps the plane solitons of the mKP-II equation into the plane solitons of the KP-II equation. More precisely, it maps the pure $2+1$-dimensional plane solitons of the mKP-II equation into the pure $2+1$-dimensional (type II) plane solitons of the KP-II equation and the singular plane solitons of the mKP-II equation into the standard (type I) plane solitons of the KP-II equation.

This last property of the Miura transformation is similar to the property of the $1+1$-dimensional Miura transformation $u_{\text{KdV}} = -(1/2)\,u_{\text{mKdV}x} - 1/4\,u^2_{\text{mKdV}}$ which, as it has been shown in [17], does not interrelates the rapidly decaying smooth and, in particular, the soliton solutions of the mKdV and KdV equations. This is quite clear from the consideration of the $1+1$-dimensional limit of the $2+1$-dimensional case. Indeed, the one-dimensional limit of the solution (6.47) ($\alpha=-\beta$) looks like

$$u_{\rm mKdV} \sim \frac{4\varepsilon}{\alpha} \frac{1}{\operatorname{sh} 2f} \qquad (7.15)$$

that is the singular solution of the mKdV equation while the 1+1-dimensional limit of the solution (7.8) ($\alpha=-\beta$) is given by (7.9). So the 1+1-dimensional Miura transformation maps the singular solutions of the mKdV-II equation into the solitons of the KdV equation.

All these indicate that the Miura transformation as the nonlinear map has a rather complicated «singularity» structure. This problem will be discussed elsewhere.

8. THE mKdV AND GARDNER EQUATIONS AS THE ONE-DIMENSIONAL LIMIT

In the usual one-dimensional limit $\partial u/\partial y=0$ the mKP equation (1.3) is reduced to the well-known mKdV equation

$$u_t + u_{xxx} - \frac{3}{2}\sigma^2 u^2 u_x = 0\,, \qquad (8.1)$$

where $\sigma^2=\pm 1$. The mKdV equation has been solved by the IST method in [32, 37] (see e. g. [2]) where solitons, breathers and multiple poles solutions have been constructed. In the one-dimensional limit $\sigma\Psi_y=-\lambda^{-2}\Psi$ and the problem (1.2a) is reduced to the one-dimensional one

$$\Psi_{xx} + V\Psi_x = -\lambda^{-2}\Psi\,. \qquad (8.2)$$

The scalar problem (8.2) is equivalent to the specialized Zakharov—Shabat matrix problem

$$\begin{pmatrix} i\partial_x, & V \\ V, & -i\partial_x \end{pmatrix} \Phi = \lambda^{-2}\Phi\,. \qquad (8.3)$$

Just this circumstance has allowed to solve the mKdV equation by the initial version of the IST method [32, 33].

The result of the previous sections give us a possibility to solve the mKdV equation, using directly the scalar problem. To do this it is sufficient to note that for the kernels R_0 of the nonlocal $\bar\partial$-problem of the form

$$R_0 = T_{1+1}(\lambda,\bar\lambda)\,\delta(\lambda'+\lambda) \qquad (8.4)$$

the dependence on y disappears and the $\bar\partial$-dressing described in section 5 gives the solutions of the mKdV equation (8.1).

In this case the problem (5.1) is reduced to the following

$$\frac{\partial\chi(x,t;\lambda,\bar\lambda)}{\partial\bar\lambda} = T_{1+1}(\lambda,\bar\lambda)\,\chi(x,t;-\lambda,-\bar\lambda)\,\exp\left(-\frac{2ix}{\lambda}-\frac{8it}{\lambda^3}\right). \qquad (8.5)$$

Wide class of solutions of the mKdV equation can be constructed with the use of the $\bar\partial$-problem (8.5).

As far as concerning the decreasing solutions and initial value problem then the combination of the constraints (8.4) and (5.15), (5.16) gives rise to the Riemann—Hilbert problem

$$\chi^+(\lambda)-\chi^-(\lambda)=\chi^-(-\lambda)\,R(\lambda)\,\exp\left(-\frac{2ix}{\lambda}-\frac{8it}{\lambda^3}\right), \quad (\operatorname{Im}\lambda=0)\,. \qquad (8.6)$$

All the solutions of the mKP equation compatible with the condition (8.4) are reduced under this constraint to the solutions of the mKdV equation. In particular, the plane solitons and breathers discussing in section 6.3 convert into the solitons and breathers of the mKdV equation [32, 33].

More general one-dimensional limit of the mKP equation arises under the constraint

$$u_y = \alpha u_x\,, \qquad (8.7)$$

where α is an arbitrary real constant.

In this case equation (1.3) converts into ($\xi=x+\alpha y$)

$$u_t + u_{\xi\xi\xi} - 3\sigma^2\left(-\alpha^2 u_\xi + \alpha u u_\xi + \frac{1}{2}u^2 u_\xi\right)=0 \qquad (8.8)$$

that is the 1+1-dimensional Gardner equation (see e. g. [1—3]).

The constraint (8.7) is equivalent to the following constraint on the kernel R of the nonlocal $\bar\partial$-problem

$$(\partial_y-\alpha\partial_x)R=0\,, \qquad (8.9)$$

i.e.

$$R = T_{1+1}(\lambda,\bar\lambda)\,\delta\left(\frac{1}{\lambda'}+\frac{1}{\lambda}-i\alpha\sigma\right). \qquad (8.10)$$

As a result, the Gardner equation (8.8) can be solved with the use of the $\bar\partial$-problem

$$\frac{\partial\chi(\xi,t;\lambda,\bar\lambda)}{\partial\bar\lambda}=T_{1+1}(\lambda,\bar\lambda)\,\chi\left(\xi,t;\frac{\lambda}{i\alpha\sigma\lambda-1}\right)e^{i\omega(\xi,t)}, \qquad (8.11)$$

where

$$\omega(\xi,t)=\left(-\frac{2}{\lambda}+i\alpha\sigma\right)\xi+4i\left(-\frac{2}{\lambda^3}+\frac{3i\alpha\sigma}{\lambda^2}+\frac{3\alpha^2\sigma^2}{\lambda}-i\alpha^3\sigma^3\right). \qquad (8.12)$$

The initial value problem and corresponding inverse problem for the Gardner-I equation ($\sigma^2=-1$) can be solved by the Riemann—Hilbert problem

$$\chi^+(\xi,t;\lambda)-\chi^-(\xi,t;\lambda)=R_0(\lambda)\,\chi^-\left(\xi,t;-\frac{\lambda}{1+\alpha\lambda}\right)e^{i\omega(\xi,t)},\quad \mathrm{Im}\,\lambda=0, \qquad (8.13)$$

which is the consequence of the constraints (8.4) and (5.15). Respectively in the case $\sigma=1$ one has the Riemann—Hilbert problem with the jump

$$\chi^+(\xi,t;\lambda)-\chi^-(\xi,t;\lambda)=R_0(\lambda)\,\chi\left(\xi,t;\frac{\lambda}{i\alpha\lambda-1}\right)e^{i\omega(\xi,t)}, \qquad (8.14)$$

across the circle $\mathrm{Im}\frac{1}{\lambda}=\frac{\alpha}{2}$ or $\lambda_R^2+\left(\lambda_I+\frac{1}{\alpha}\right)^2=\frac{1}{\alpha^2}$.

Note that for the Gardner-I ($\sigma=i$) and the Gardner-II ($\sigma=1$) equations we have the different Riemann—Hilbert problems (8.13) and (8.14) in contrast to the unique Riemann—Hilbert problem (8.6) for the mKdV-I ($\sigma=i$) and mKdV-II ($\sigma=1$) equations.

9. CONCLUSION

We see that the mKP equation is solvable, generally, by the same IST method as the KP equation. These two equations are, in fact, closely interrelated. As we discussed in section 7, they are connected by the Miura transformation [23, 24].

The KP equation has the solutions with the functional parameters which can be represented in the form [1]

$$u_{KP}=2\frac{\partial^2}{\partial x^2}\ln\det A, \qquad (9.1)$$

where

$$A_{nm}=\delta_{nm}+\int_{-\infty}^{x}dx'\,h_n(x',y,t)\,g_m(x',y,t)$$

and $h_n(x,y,t)$ and $g_m(x,y,t)$ are the complex-valued solutions of the linearized KP equation.

The linearized KP and mKP equations coincide. So, the same set of arbitrary solutions of the linear equation $h_t+h_{xxx}+3\sigma^2\partial_x^{-1}h_{yy}=0$ gives rise to the solutions both of the KP and mKP equations via the formulae (9.1) and (1.7).

The $\bar\partial$-dressing method reveals even more deep interrelation between the KP and mKP equations. Indeed, the KP equation can be constructed with the use of the $\bar\partial$-problem (5.1)—(5.3) but with the normalization $\chi\xrightarrow[\lambda\to 0]{}\lambda^{-1}$ [22]. So within the framework of the $\bar\partial$-dressing method the only difference between the KP and mKP equations consists in the different normalization of χ. In more detail this interrelation between the KP and mKP equations will be considered elsewhere.

Appendix A

Alternative Way of Introducing the Spectral Parameter

As we have seen the introduction of the spectral parameter λ into the problem (2.1a) via (2.2) leads to the function χ which admits the canonical normalization $\chi\xrightarrow[\lambda\to\infty]{}1$. Such a normalization is crucial for the derivation of the inverse problem equations with the constant inhomogeneous term 1.

But there exist, of course, another way of introducing the spectral parameter λ which is exactly the same as for the KP equation. It is

$$\Psi(x,y)=\mu(x,y;\lambda)\,e^{i(\lambda x-\lambda^2 y)}. \qquad (A.1)$$

The corresponding equation for μ is

$$i\mu_y+\mu_{xx}+2i\lambda\mu_x+u(i\partial_x-\lambda)\mu=0. \qquad (A.2)$$

The Green functions for equation (A.2) are differed from the Green functions for (2.3) only by the substitution $\lambda\to\lambda^{-1}$. Hence, again one is able to construct the Green functions G^+ and G^- for (A.2) which are analytic in upper and lower half planes and, consequently, define the solutions μ^+ and μ^- of (A.2) which have the same analytic properties. Then, repeating the construction of section 2, one arrives again at the singular nonlocal Riemann—Hilbert problem.

The essential difference with section 2 is that now the functions $\mu^{\pm}$ do not admit the canonical normalizations. Indeed, the substitution of the asymptotic expansion $\mu \sim \mu_{\infty} + (1/\lambda)\mu_{-1} + \dots$ at $\lambda \to \infty$, into (A.2) gives

$$2i\mu_{\infty x} = u(x, y, t)\,\mu_{\infty}\,. \tag{A.3}$$

So μ_{∞} is the functional on $u(x, y, t)$. Hence, the corresponding inverse problem equations will contain the inhomogeneous term μ_{∞} which depends on «potential» $u(x, y)$ itself. This makes the solvability problem of the inverse problem equations very complicated.

One can bypass this problem by transiting to the new function $\tilde{\mu}$ defined by

$$\tilde{\mu}(x, y; \lambda) = \frac{\mu(x, y; \lambda)}{\mu_{\infty}(x, y)}\,. \tag{A.4}$$

The function $\tilde{\mu}(x, y; \lambda)$ has the canonical normalization and its jump is given by the relation (2.23) with substitution $\mu \to \tilde{\mu}$. For such function $\tilde{\mu}$ the inverse problem equations are of the form (2.26), (2.27), i.e. they contains the constant inhomogeneous term. So there is no problem with their solvability.

The reconstruction formula now is

$$u(x, y) = -2i\frac{\partial}{\partial x}\ln \tilde{\mu}(x, y; 0) \tag{A.5}$$

since one can choose $\tilde{\mu}(x, y; 0) = 1$.

It is not difficult to see that the results one can obtain in this approach are equivalent to those of section 2. Nevertheless, the approach of section 2 is preferable and more adequate to the problem since it does not contain any intermediate functions, does not required any constraint on the inverse problem data and reveals a deep connection with the KP equation within the $\bar{\partial}$-dressing method.

Appendix B

Identities for the Discrete Spectrum

To derive the relation (2.28) we, similar to the KP case, introduce the functions

$$\hat{\chi}^{\pm}(x, y; \lambda) \overset{def}{=} \chi^{\pm}(x, y; \lambda) \exp \tilde{\omega}(x, y; \lambda) \tag{B.1}$$

and

$$\hat{\chi}_k^{\pm}(x, y) \overset{def}{=} \chi_k^{\pm}(x, y) \exp \tilde{\omega}(x, y; \lambda_k^{\pm})\,. \tag{B.2}$$

They obey the integral equations

$$\hat{\chi}^{\pm}(x, y; \lambda) - \left[\tilde{G}^{\pm}(\cdot\,; \lambda)\, u(\cdot) \left(i\partial' - \frac{1}{\lambda}\right) \hat{\chi}^{\pm}(\cdot\,; \lambda)\right](x, y) = 0 \tag{B.3}$$

and

$$\hat{\chi}_k^{\pm}(x, y) - \left[\tilde{G}^{\pm}(\cdot\,; \lambda_k^{\pm})\, u(\cdot) \left(i\partial' - \frac{1}{\lambda_k^{\pm}}\right) \hat{\chi}_k^{\pm}(\cdot)\right](x, y) = 0\,, \tag{B.4}$$

where

$$(\tilde{G}^{\pm}(\cdot\,; \lambda) f)(x, y) \overset{def}{=} \exp(\tilde{\omega}(x, y; \lambda))\,(G^{\pm}(\cdot\,; \lambda) \exp(-\tilde{\omega}(\cdot\,; \lambda)) f)(x, y) \tag{B.5}$$

and the functions $G^{\pm}$ are given by (2.6).

Equations (B.3) and (B.4) imply that the function

$$\hat{\Phi}_k^{\pm}(x, y; \lambda) \overset{def}{=} \left(\chi^{\pm}(x, y; \lambda) - \frac{c_k^{\pm} \chi_k^{\pm}(x, y)}{\lambda - \lambda_k^{\pm}}\right) \exp[\tilde{\omega}(x, y; \lambda)] \tag{B.6}$$

obeys the following integral equation:

$$\hat{\Phi}_k^{\pm}(x, y; \lambda) - \left[\tilde{G}^{\pm}(\cdot\,; \lambda)\, u(\cdot)\left(i\partial' - \frac{1}{\lambda}\right) \hat{\Phi}_k^{\pm}(\cdot\,; \lambda)\right](x, y) =$$

$$= \exp(\tilde{\omega}(x, y; \lambda)) - \frac{c_k^{\pm}}{\lambda - \lambda_k^{\pm}}\left[\hat{\chi}_k^{\pm} - \tilde{G}^{\pm}(\cdot\,; \lambda)\, u\left(i\partial' - \frac{1}{\lambda}\right)\hat{\chi}^{\pm}\right](x, y)\,. \tag{B.7}$$

Proceeding in (B.7) to the limits $\lambda \to \lambda_k^{\pm}$, one obtains

$$\hat{\Phi}_k^{\pm}(x, y; \lambda_k^{\pm}) - \left[\tilde{G}^{\pm}(\cdot\,; \lambda_k^{\pm})\, u(\cdot)\left(i\partial' - \frac{1}{\lambda_k^{\pm}}\right) \hat{\Phi}_k^{\pm}(\cdot\,; \lambda_k^{\pm})\right](x, y) =$$

$$= \exp(\tilde{\omega}(x, y; \lambda_k^{\pm})) - c_k^{\pm}\left[\frac{\partial}{\partial\lambda}\left(\tilde{G}^{\pm}(\cdot\,; \lambda) u(\cdot)\left(i\partial' - \frac{1}{\lambda}\right)\right)\Big|_{\lambda=\lambda_k^{\pm}} \hat{\chi}_k^{\pm}(\cdot)\right](x, y) -$$

$$- c_k^{\pm}\left[\frac{\partial\hat{\chi}^{\pm}}{\partial\lambda}\Big|_{\lambda=\lambda_k^{\pm}} - \tilde{G}^{\pm}(\cdot\,; \lambda_k^{\pm})\, u(\cdot)\left(i\partial' - \frac{1}{\lambda_k^{\pm}}\right) \frac{\partial\hat{\chi}^{\pm}}{\partial\lambda}\Big|_{\lambda=\lambda_k^{\pm}}\right](x, y)\,. \tag{B.8}$$

It follows from (B.5) that

$$\left[\frac{\partial(\tilde{G}^{\pm}(\cdot\,; \lambda)\, u(\cdot)\,(i\partial' - 1/\lambda))}{\partial\lambda}\Big|_{\lambda=\lambda_k^{\pm}} f(\cdot)\right](x, y) =$$

$$= \mp \frac{1}{2\pi\lambda_k^{\pm 2}} \int\limits_{-\infty}^{+\infty}\!\!\int d\eta \exp[\tilde{\omega}(x-\xi, y-\eta; \lambda_k^{\pm})]\, u(\xi,\eta)\, \partial_\xi f(\xi,\eta)\,. \qquad (B.9)$$

In virtue of (B.9) equation (B.8) is equivalent to the following

$$\left[\left(1-\tilde{G}^{\pm}(\cdot;\lambda_k^{\pm})\, u(\cdot)\left(i\partial' - \frac{1}{\lambda_k^{\pm}}\right)\right)\left(\hat{\Phi}_k^{\pm}\Big|_{\lambda=\lambda_k^{\pm}} + c_k^{\pm}\frac{\partial\hat{\chi}^{\pm}}{\partial\lambda}\Big|_{\lambda=\lambda_k^{\pm}}\right)\right](x,y) =$$

$$= \exp[\tilde{\omega}(x,y;\lambda_k^{\pm})]\left[1 \pm \frac{c_k^{\pm}}{2\pi\lambda_k^{\pm 2}} \int\limits_{-\infty}^{+\infty}\!\!\int d\eta\, u(\xi,\eta)\left(\partial_\xi + \frac{i}{\lambda_k^{\pm}}\right)\chi_k^{\pm}(\xi,\eta)\right]. \qquad (B.10)$$

We assume that the singular points for equation (2.8) are non-degenerated. As a result, the Fredholm alternative implies

$$1 \pm \frac{c_k^{\pm}}{2\pi\lambda_k^{\pm 2}} \int\limits_{-\infty}^{+\infty}\!\!\int d\eta\, u(\xi,\eta)\left(\partial_\xi + \frac{i}{\lambda_k^{\pm}}\right)\chi_k^{\pm}(\xi,\eta) = 0 \qquad (B.11)$$

and

$$\hat{\Phi}_k^{\pm}|_{\lambda=\lambda_k^{\pm}} + c_k^{\pm}\frac{\partial\hat{\chi}^{\pm}}{\partial\lambda}\Big|_{\lambda=\lambda_k^{\pm}} = \gamma_k \hat{\chi}_k^{\pm}(x,y)\,, \qquad (B.12)$$

where γ_k are some constants. It is easy to see that (B.12) is equivalent to the relation

$$\lim_{\lambda\to\lambda_k^{\pm}}\left(\chi^{\pm}(x,y;\lambda) - \frac{c_k^{\pm}\chi_k^{\pm}(x,y)}{\lambda-\lambda_k^{\pm}}\right) = \left(\frac{ic_k^{\pm}}{\lambda_k^{\pm 2}}\left(x - \frac{2y}{\lambda_k^{\pm}}\right) + \gamma_k^{\pm}\right)\chi_k^{\pm}\,. \qquad (B.13)$$

The constants $c_k^{\pm}$ are fixed by the normalization of $\chi_k^{\pm}(x, y)$. Integral equations (B.3) and (B.4) imply that

$$\chi^{\pm}(x,y;\lambda) \xrightarrow[(x^2+y^2)^{1/2}\to\infty]{} 1 \qquad (B.14)$$

and

$$\chi_k^{\pm}(x,y) \xrightarrow[(x^2+y^2)^{1/2}\to\infty]{} \frac{\alpha}{x-(2y/\lambda_k^{\pm})}\,, \qquad (B.15)$$

where α is some constant. So the functions $\chi_k^{\pm}(x, y)$ admit the normalization

$$\lim_{(x^2+y^2)^{1/2}\to\infty}\left(x - \frac{2y}{\lambda_k^{\pm}}\right)\chi_k^{\pm}(x,y) = 1\,. \qquad (B.16)$$

Considering the relation (B.13) at $(x^2+y^2)^{1/2}\to\infty$ and taking into account (B.14) and (B.16), one gets

$$c_k^{\pm} = -i\lambda_k^{\pm 2}\,. \qquad (B.17)$$

With such $c_k^{\pm}$ the relation (B.13) coincides with (2.28).

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