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NONLINEAR COHERENT BEAM-BEAM OSCILLATIONS IN THE RIGID BUNCH MODEL

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Nonlinear Coherent Beam-Beam
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In the Rigid Bunch Model

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A B S T R A C T

Within the framework of the rigid bunch model coherent oscillations of strong-strong colliding bunches are described by equations which are specific for the weak-strong beam case. In this paper some predictions of the model for properties of nonlinear coherent oscillations as well as for associated limitations of the luminosity are discussed.

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1. INTRODUCTION

It is well known, that the luminosity of a collider can be limited by the beam-beam interaction — the distortion of the particles' motion due to fields of the counter-moving beam. In fact this phenomenon is very complicated and many effects can be responsible for the limitation of the luminosity in particular conditions. Therefore, some simplifying assumptions should be used to get definite predictions concerning the beam-beam interaction. One of the widely used approaches, based on the so-called weak-strong beam approximation. Within the framework of this approach the motion of one particle from the weak beam is traced under perturbations due to given fields of the counter-moving strong beam. The periodicity and strong nonlinearity of such perturbations manifest the importance of nonlinear resonances for the beam-beam instability in this case.

In spite of very important predictions the weak-strong model obviously gives a limiting view on the problem. Once the interaction perturbs the motion of particles in both colliding beams they evolve in a self-consistent way — the disturbance of the beam changes the fields, which disturb the counter-moving beam. Such a self-consistent behaviour of colliding beams becomes especially important in the case of a strong-strong interaction and gives additional effects, which must be described by the theory. One of the simplest and most important problems is that of collective stability of colliding beams [1—5]. It can be treated in the linear approximation on amplitudes of coherent oscillations and hence can be solved exactly. The employment of various techniques predicts both suitable positions for the working point of the machine and thresholds of instabilities.

Nevertheless both experimental results and multiparticle tracking (see, for instance, Refs [6—8] definitely indicate the importance of theoretical study of nonlinear coherent phenomena. In fact, this is many-fold problem, which is directly related to the behaviour of colliding beams on large times and therefore to limitations of the ring luminosity. Since general description of nonlinear coherent phenomena is out of analytical methods, in this paper we shall consider the description of nonlinear coherent oscillations of colliding beams within the framework of the so-called rigid bunch model. This means that we shall interest in the behaviour of coherent oscillations, which are far away from the statistical equilibrium. Due to its simplicity the rigid bunch model is frequently used to study coherent beam-beam effects. Recently it was done in Ref. [9] for the calculation of nonlinear corrections to the beam-beam coherent tune shift for the beam parameters diagnostic based on the measurement of beam response spectra. The exitation of nonlinear beam-beam resonances can disturb results of such measurements. Initial calculations concerning that limitations were done in Ref. [10] and will be discussed in this paper in more details. Computer simulations in Refs [7, 8] actually used the fields in the form specific for this model.

2. GENERAL EQUATIONS

In the rigid bunch model coherent oscillations of a bunch are described by the displacement of its distribution function $f(\vec{r}, \vec{p}, t)$ as a whole. If $\vec{F}$ is the total force acting on a particle and the evolution of the distribution function is governed by the Vlasov's equation:

$$\frac{\partial f}{\partial t} + \vec{v}\,\frac{\partial f}{\partial \vec{r}} + \vec{F}\,\frac{\partial f}{\partial \vec{p}} = 0, \tag{1}$$

one can easily obtain the equations:

$$\frac{d}{dt}\langle \vec{r}\,\rangle = -\int d\Gamma\,\vec{r}\left(\vec{v}\frac{\partial f}{\partial \vec{r}} + \vec{F}\frac{\partial f}{\partial \vec{p}}\right) = \langle \vec{v}\rangle, \qquad d\Gamma = d^3p\,d^3r,$$

$$\frac{d}{dt}\langle \vec{p}\rangle = -\int d\Gamma\,\vec{p}\left(\vec{v}\frac{\partial f}{\partial \vec{r}} + \vec{F}\frac{\partial f}{\partial \vec{p}}\right) = \langle \vec{F}\rangle, \tag{2}$$

for the dipole moments of $f(\vec{r}, \vec{p}, t)$:

$$\langle \vec{r}\rangle = \int d\Gamma\,\vec{r} f(\vec{r}, \vec{p}, t),$$

$$\langle \vec{p}\rangle = \int d\Gamma\,\vec{p} f(\vec{r}, \vec{p}, t), \tag{3}$$

$\vec{p}$ is the momentum of a particle.

Since $\langle \vec{F}\rangle$ generally depends on the higher order momenta of the distribution function

$$\langle \vec{F}\rangle = \vec{F}(0) + \frac{\partial \vec{F}}{\partial r_\alpha}\langle r_\alpha\rangle + \frac{1}{2}\,\frac{\partial^2 \vec{F}}{\partial r_\alpha \partial r_\beta}\langle r_\alpha r_\beta\rangle + \ldots$$

eqs (2) are not closed. For instance, we have ($p_z = \gamma m v_z$, γ is relativistic factor):

$$\frac{d}{dt}\langle z^2\rangle = 2\langle z v_z\rangle; \qquad \frac{d}{dt}\langle z v_z\rangle = \langle v_z^2\rangle + \frac{1}{\gamma m}\langle z F_z\rangle;$$

$$\frac{d}{dt}\langle v_z^2\rangle = \frac{2}{\gamma m}\langle v_z F_z\rangle, \tag{4}$$

and therefore

$$\frac{d}{dt}\sigma_z^2 = \frac{d}{dt}\langle z^2\rangle - 2\langle z\rangle\frac{d}{dt}\langle z\rangle = 2(\langle z v_z\rangle - \langle z\rangle\langle v_z\rangle),$$

$$\frac{d}{dt}\sigma_{p_z}^2 = \frac{d}{dt}\langle p_z^2\rangle - 2\langle p_z\rangle\frac{d}{dt}\langle p_z\rangle = 2(\langle p_z F_z\rangle - \langle p_z\rangle\langle F_z\rangle).$$

Nevertheless, if the distribution function has the form, specific for the rigid bunch approximation

$$f(\vec{r}, \vec{p}, t) = f_0(\vec{r} - \langle \vec{r}(t)\rangle;\ \ \vec{p} - \langle \vec{p}(t)\rangle) \tag{5}$$

σ_z^2 and $\sigma_{p_z}^2$ (as well as higher order spreads) will be conserved, whereas higher order momenta of $f(\vec{r}, \vec{p}, t)$ can be calculated via $\langle \vec{z}\rangle$, $\langle \vec{v}_z\rangle$ and those constant spreads. Therefore, the rigid bunch model by means of eqs (2) describes coherent oscillations of the bunch with constant sizes and shape. It is certainly clear that the distribution (5) is quite unrealistic. Due to the nonlinearity of forces disturbing the motion of particles, once centered around $\langle z(t=0)\rangle$ and $\langle p_z(t=0)\rangle$, the distribution will dilute in phases of oscillations with corresponding decay of $\langle z\rangle$, $\langle p_z\rangle$ and enlargement of the bunch effective emittance. This means that the model will give an adequate description of coherent oscillations only during the time intervals, which are limited

$$\Delta t \ll 1/\Delta\omega$$

by the frequency spread in the beam $\Delta\omega$ (or, probably, by the rise time of the instability).

If we shall describe the motion of a particle in the storage ring in the smooth approximation, the force $F_{x,z}$ can be written in the form

$$F_x = -\gamma m\omega_x^2 x + \delta F_x\,, \qquad F_z = -\gamma m\omega_z^2 z + \delta F_z\,,$$

where $E=\gamma mc^2$ is the particle energy and $\omega_{x,z}=\omega_0\nu_{x,z}$ are the frequencies of betatron oscillations. Hence, in the rigid bunch model coherent betatron oscillations are described by the following (well known) equations:

$$\langle\ddot{x}\rangle + \omega_x^2\langle x\rangle = \frac{1}{\gamma m}\langle\delta F_x\rangle,$$

$$\langle\ddot{z}\rangle + \omega_z^2\langle z\rangle = \frac{1}{\gamma m}\langle\delta F_z\rangle. \qquad (6)$$

Let us now apply eqs (6) to the description of betatron coherent oscillations of colliding beams. For the sake of simplicity we shall assume that two relativistic ($\gamma\gg1$) bunches with densities $N_1\rho^{(1)}$ and $N_2\rho^{(2)}$ move in the same ring and collide at one interaction point [IP]. The force distorting the motion of particles from the counter-moving beam is determined by the Lagrangian:

$$\delta\vec{F}^{(1,2)} = \frac{\partial}{\partial\vec{r}}L_{1,2}\,, \qquad (7)$$

$$L_{1,2} = \frac{N_2e^2\delta_T(t)}{c}U_{1,2}(\vec{r}_\perp,t)\,, \qquad e_1e_2=-e^2;$$

$$\delta_T(t) = \sum_{k=-\infty}^{\infty}\delta\left(t-\frac{2\pi k}{\omega_0}\right), \qquad (8)$$

$$U_{1,2} = \int\frac{d^2k}{\pi k^2}\exp(i\vec{k}\vec{r}_\perp)\,\rho^{(2)}(\vec{k},t)\,,$$

$$\rho^{(2)}(\vec{k},t) = \int d^2r_\perp\exp(-i\vec{k}\vec{r}_\perp)\,\rho^{(2)}(\vec{r}_\perp,t)\,. \qquad (9)$$

Once the model describes the oscillations of the bunch by the distribution function (5), instead of (9) one has

$$\rho^{(2)}(\vec{k}) = \exp(-i\vec{k}\langle\vec{r}_\perp(t)\rangle)\,\rho_0^{(2)}(\vec{k})\,,$$

and therefore [9]:

$$\langle\delta\vec{F}^{(1,2)}\rangle = \int d^3r\,\rho_0^{(1)}(\vec{r}_\perp-\langle\vec{r}_\perp^{(1)}\rangle)\frac{\partial}{\partial\vec{r}_\perp}L_{1,2} =$$

$$= \frac{N_2e^2\delta_T(t)}{c}\int\frac{d^2k}{\pi k^2}\rho_0^{(2)}(\vec{k})\,\rho_0^{(1)}(-\vec{k})\,i\vec{k}\exp[i\vec{k}(\langle r_\perp^{(1)}\rangle-\langle\vec{r}_\perp^{(2)}\rangle)] =$$

$$= \frac{N_2e^2\delta_T(t)}{c}\frac{\partial}{\partial\vec{b}}\int\frac{d^2k}{\pi k^2}\exp(i\vec{k}\vec{b})\,\rho_0^{(2)}(\vec{k})\,\rho_0^{(1)}(-\vec{k})\,, \qquad (10)$$

$$\vec{b} = \langle\vec{r}_\perp^{(1)}(t)\rangle - \langle\vec{r}_\perp^{(2)}(t)\rangle.$$

Using eq. (10) we can rewrite eqs (6) for, say, vertical betatron coherent oscillations in the following form:

$$\langle\ddot{z}^{(1)}\rangle + \omega_z^2\langle z^{(1)}\rangle = \frac{N_2e^2\delta_T(t)}{\gamma mc}\frac{\partial}{\partial b_z}U_{1,2}(b)\,, \qquad (12)$$

$$\langle\ddot{z}^{(2)}\rangle + \omega_z^2\langle z^{(2)}\rangle = -\frac{N_1e^2\delta_T(t)}{\gamma mc}\frac{\partial}{\partial b_z}U_{2,1}(b)\,, \qquad (13)$$

$$U_{1,2} = \int\frac{d^2k}{\pi k^2}\exp(i\vec{k}\vec{b})\,\rho_0^{(1)}(-\vec{k})\,\rho_0^{(2)}(\vec{k})\,. \qquad (14)$$

From eqs (12), (13), one can see that for beams moving in the same ring the beam-beam interaction affects only the relative motion of colliding beams (π-mode). Such oscillations are described by the impact parameter $\vec{b}$, which satisfies the following equation:

$$\ddot{b}_z + \omega_z^2 b_z = \frac{2Ne^2}{\gamma mc}\delta_T(t)\frac{\partial}{\partial b_z}U(\vec{b})\,, \qquad (15)$$

where

$$U(\vec{b}) = \frac{1}{2N}[N_2U_{1,2} + N_1U_{2,1}]\,, \qquad N=(N_1+N_2)/2\,. \qquad (16)$$

It is remarkable that eq. (15) describing the interaction of two generally strong-strong colliding bunches has exactly the same form as the equation of motion for single particle in the weak-strong beam approximation (but with the special distribution of particles

$$\rho_{eff}^{(k)} = \frac{1}{2N}[N_2\rho_0^{(1)}(-\vec{k})\,\rho_0^{(2)}(\vec{k}) + N_1\rho_0^{(2)}(-\vec{k})\,\rho_0^{(1)}(\vec{k})]$$

in the strong beam). Hence, all the results of the weak-strong theory can be applied for the description of coherent oscillations of strong-strong beams within the framework of the rigid bunch model. In particular, if $\rho^{(1,2)}$ are Gaussian distributions

$$\rho_0^{(1)}(k)=\exp\left(-\frac{k_x^2\sigma_{1x}^2}{2}-\frac{k_z^2\sigma_{1z}^2}{2}\right),$$

$$\rho_0^{(2)}(k)=\exp\left(-\frac{k_x^2\sigma_{2x}^2}{2}-\frac{k_z^2\sigma_{2z}^2}{2}\right),$$

$\rho_{eff}^{(k)}$ is also Gaussian [9]

$$\rho_{eff}(k)=\exp\left(-\frac{k_x^2\Sigma_x^2}{2}+\frac{k_z^2\Sigma_z^2}{2}\right),$$

$$\Sigma_x^2=\sigma_{1x}^2+\sigma_{2x}^2,\qquad \Sigma_z^2=\sigma_{1z}^2+\sigma_{2z}^2$$

and the results for the strong-strong case can be obtained from weak-strong calculations using simple scale transformation.

Let us note also that eq. (15) is generated by the Hamiltonian:

$$H=\frac{1}{2}\left[b_z'^2+\omega_z^2 b_z^2\right]-\frac{2Ne^2}{\gamma mc}\,\delta_T(t)\,U(\bar{b})\,. \qquad (17)$$

The unperturbed coherent oscillations are described by the following formulae

$$b_z=(2J\bar{\beta})^{1/2}\cos\psi\,,\qquad b_z'=db/d\theta=-\nu_z(2J/\bar{\beta})^{1/2}\sin\psi\,;$$

$$\psi'=\nu_z\,,\qquad \theta=\omega_0 t\,,\qquad \bar{\beta}=R_0/\nu_z\,, \qquad (18)$$

which generate the canonical transformation from variables (b, b') to action-phase variables (J, ψ). In these variables the Hamiltonian takes the form:

$$H(J,\psi)=\nu_z J-\frac{2Nr_0}{\gamma}\,\delta_T(\theta)\,U(\bar{b})\,, \qquad (19)$$

where $r_0=e^2/mc^2$ and $2\pi R_0$ is the perimeter of the orbit.

3. COHERENT OSCILLATIONS WITH SMALL AMPLITUDES

First, let us consider the case, when colliding beams have small vertical coherent oscillations. We shall assume that the densities in

both beams are Gaussian:

$$\rho_0^{(1,2)}(x,z)=\frac{1}{2\pi\sigma_x\sigma_z}\exp\left[-\frac{x^2}{2\sigma_x^2}-\frac{z^2}{2\sigma_z^2}\right]. \qquad (20)$$

The Hamiltonian (19) takes the form:

$$H(J,\psi)=\nu_z J-\frac{2Nr_0}{\gamma}\,\delta_T(\theta)\,U(b)\,,$$

$$U(b_z)=\int_0^\infty\frac{ds}{\sqrt{(s+\sigma_x^2)(s+\sigma_z^2)}}\exp\left(-\frac{b_z^2}{4[s+\sigma_z^2]}\right). \qquad (21)$$

For small oscillations $|b_z|\ll\sigma_z$ factor U can be expanded into the series over the powers of b_z. In the lowest order approximation one has

$$U(b_z)\simeq-\frac{b_z^2}{2}\,\frac{1}{\sigma_z(\sigma_x+\sigma_z)}$$

The substitution of this expression into (21) yields

$$H(J,\psi)=\nu_z J+\xi_z J(1+\cos 2\psi)\sum_{n=-\infty}^{\infty}\exp(-in\theta)\,, \qquad (22)$$

where

$$\xi_z=\frac{Nr_0\bar{\beta}}{2\pi\gamma\sigma_z(\sigma_x+\sigma_z)} \qquad (23)$$

is the beam-beam parameter for vertical coherent oscillations. Here it coincides with that for individual particles [9]. The twice large value for ξ_z can be obtained after linearization of motion equations for individual particles over $\vec{r}^{(a)}-\langle\vec{r}^{(b)}\rangle$ and subsequent averaging of these equations. The same result can be also obtained by the direct solution of linearized Vlasov's equations [3, 5, 6]. This discrepancy is caused by the contribution (22) from the nonlinear part of the force acting on a particle at the (IP) [10].

From (22) one can deduce that coherent oscillations of colliding beams have nontrivial behaviour provided the tune ν_z is a close to the resonant value $\nu_z=n/2$ $(n=1, 2, \dots)$. In the resonant case the usage of the first approximation of the averaging method [11] gives new Hamiltonian, which in slow variables J and $\varphi=\psi-n/2$ has the form

$$H(J,\varphi)=\Delta J+J\xi_z(1+\cos 2\varphi)\,,\qquad \Delta=\nu-n/2\,. \tag{24}$$

The main features of the motion described by this Hamiltonian are very simple. Due to the conservation of H, along the phase trajectory J changes from the line (see Fig. 1)

$$H_+\equiv H(\cos 2\varphi=1)=(\Delta+2\xi_z)\,J \tag{25}$$

to the line

$$H_-\equiv H(\cos 2\varphi=-1)=\Delta\cdot J \tag{26}$$

and back. As can be seen from Fig. 1,*a*,*b*,*c* this motion will be unstable provided

$$\Delta<0\,,\qquad |\Delta|<2\xi_z\,, \tag{27}$$

which corresponds to open Hamiltonians $H_\pm$ (see Fig. 1,*c*). This stability condition certainly argees with that obtained by direct solution of linearized eq. (15). Since in this approach the beam-beam parameters for coherent oscillations and for the motion of individual particles coincide, the motion of individual particles becomes unstable exactly inside the stopbands (27). This fact obviously breaks the initial assumption, that the beam distributions are unaltered, at least, during the rise time of the instability. The contradiction is caused by the mentioned contribution into (22) from nonlinearities of the beam-beam forces. Hence more consistent calculations within the framework of the rigid bunch model should take into account nonlinear dependence of Hamiltonian (22) on b_z.

For small coherent oscillations the first nonlinear correction to (22) is determined by

$$\delta U=\frac{b_z^4}{48}\,\frac{\varkappa}{\sigma_z^3(\sigma_x+\sigma_z)}\,,\qquad \varkappa=\frac{\sigma_x+2\sigma_z}{\sigma_x+\sigma_z} \tag{28}$$

the cubic nonlinearity of the beam-beam force. It is well known that generally such a nonlinearity stabilizes resonances up to the 4th order. For resonances $\nu_z\simeq n/2$, (28) modifies the Hamiltonian (22) in the following way:

$$H=J\,[\,\Delta+\xi(1+\cos 2\varphi)]-\frac{\xi_z\varkappa}{64\varepsilon_z}\,J^2\,,\qquad \sigma_z^2=2\varepsilon_z\beta_z\,. \tag{29}$$

It is clear from (29), that nonlinear part closes Hamiltonians $H_\pm$, which causes the stabilization of oscillations. Nevertheless, due to very weak dependence of H on J^2, it is clear also that the model

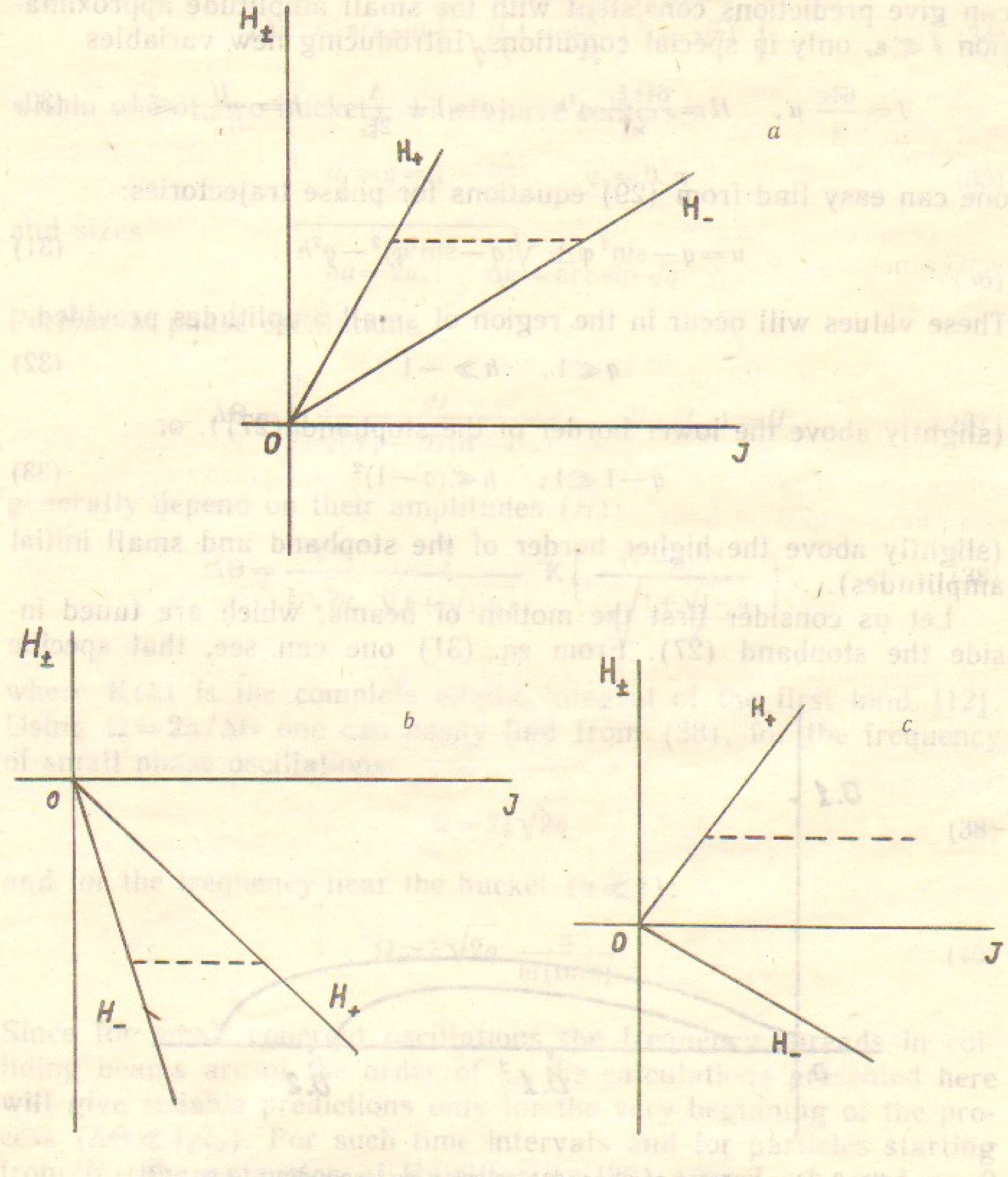

Fig. 1. Hamiltonians $H_\pm$ for resonances $\nu=n/2$: $\Delta>0$ (*a*); $\Delta<-2\xi$ (*b*); $-2\xi<\Delta<0$ (*c*).

can give predictions consistent with the small amplitude approximation $J \ll \varepsilon_z$ only in special conditions. Introducing new variables

$$J = \frac{64\varepsilon}{\varkappa} u, \quad H = \frac{64\varepsilon\,\xi_z}{\varkappa} q^2 h; \quad q = 1 + \frac{\Delta}{2\xi_z}, \quad h = \frac{H}{H_+^{\max}} \leqslant 1, \tag{30}$$

one can easy find from (29) equations for phase trajectories:

$$u = q - \sin^2\varphi \pm \sqrt{(q - \sin^2\varphi)^2 - q^2 h}\ . \tag{31}$$

These values will occur in the region of small amplitudes provided

$$q \ll 1, \quad h \gg -1 \tag{32}$$

(slightly above the lower border of the stopband (27)), or

$$q - 1 \ll 1, \quad h \ll (q-1)^2 \tag{33}$$

(slightly above the higher border of the stopband and small initial amplitudes).

Let us consider first the motion of beams, which are tuned inside the stopband (27). From eq. (31) one can see, that specific

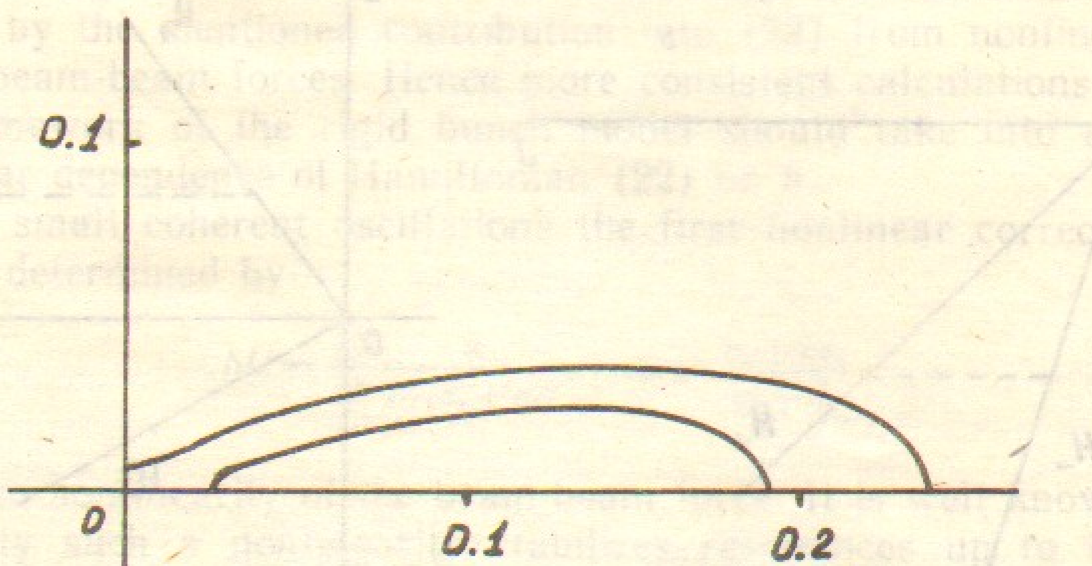

Fig.2. One quarter of phase trajectories corresponding to eq. (31): *1*—oscillation captured into the buchet, $q=0.1$, $h=0.1$; *2*—oscillations with infinite motion in phase φ, $q=0.1$, $h=-1$.

features of such coherent oscillations strongly depend on the sign of h. Namely, if $h<0$, oscillations of J are accompanied by the infinite increase of the slow phase φ (see Fig. 2).

In contrast to that, if $h>0$ the phase φ makes finite oscillations

$$|\Delta\varphi| \leqslant \arcsin\left\{\sqrt{\left(1 - \frac{|\Delta|}{2\xi}\right)(1 - \sqrt{h})}\right\} \tag{34}$$

within one of two buckets, which have centers at

$$u_s = q = 1 - \frac{|\Delta|}{2\xi}, \quad \varphi_s = 0, \pi \tag{35}$$

and sizes

$$\delta u = 2u_s; \quad \Delta\varphi = \arcsin\sqrt{q}\,. \tag{36}$$

Periods of phase oscillations

$$\Delta\Theta = \int_{J_-}^{J_+} \frac{dJ}{\sqrt{(H_+ - H)(H - H_-)}}, \quad H_+(J_\pm) = H, \tag{37}$$

generally depend on their amplitudes (H):

$$\Delta\Theta = \frac{1}{\xi_z\sqrt{2q}} \frac{2}{\sqrt{1+\sqrt{1-h}}} \mathbf{K}\left\{\frac{(1-h)^{1/4}}{\sqrt{\frac{1+\sqrt{1-h}}{2}}}\right\}, \tag{38}$$

where $\mathbf{K}(k)$ is the complete elliptic integral of the first kind [12]. Using $\Omega = 2\pi/\Delta\Theta$ one can easily find from (38), for the frequency of small phase oscillations:

$$\Omega = 2\xi\sqrt{2q} \tag{38}$$

and for the frequency near the bucket ($h \ll 1$):

$$\Omega \simeq \xi\sqrt{2q}\ \frac{\pi}{\ln(16/h)}. \tag{40}$$

Since for small coherent oscillations the frequency spreads in colliding beams are of the order of ξ_z, the calculations presented here will give reliable predictions only for the very beginning of the process ($\Delta\Theta \ll 1/\xi_z$). For such time intervals and for particles starting from H_+ the expansion of Hamiltonian (29) near $J = J_-$ and $\varphi = 0$ in the lowest order approximation yields the suppression of increments:

$$\delta \simeq \sqrt{2\xi_z|\Delta|\left(1 - \frac{|\Delta|}{2\xi} - \frac{\varkappa J_-}{64\varepsilon_z}\right)} \tag{41}$$

due to the nonlinearity of the beam-beam interaction. Such a saturation of increments by the finite values of amplitudes is specific for coherent oscillations with weak turbulence (see, for instance, [13]).

Except for the stabilization of resonances $\nu_z=n/2$, the cubic nonlinearity of the beam-beam force also generates resonances $\nu_z=n/4$. We shall not discuss this case in detail. Let us note only that it gives no unstable solutions and that the model predicts for such resonances coherent oscillations with small amplitudes ($J\ll\varepsilon_z$) provided $|\Delta+\xi_z|\ll\xi_z$.

4. LARGE AMPLITUDES

To simplify the calculations for coherent oscillations with large amplitudes ($J\gtrsim\varepsilon$) we shall use the model, which assumes the special distributions in both beams

$$\rho^{(1,2)}(x,z)=\frac{N\delta(z)}{\sqrt{2\pi}\,\sigma}\exp\left(-\frac{x^2}{2\sigma^2}\right) \qquad (42)$$

and that only horizontal coherent oscillations are excited in beams ($b_z=0$). Under these assumptions coherent oscillations of colliding beams near a resonance $\nu_x=n/m$ are described by the following Hamiltonian:

$$H^{(m)}(J,\varphi)=\begin{cases}\Delta_m J+4\xi\varepsilon U_0\left(\frac{J}{4\varepsilon}\right)-(-1)^{m/2}8\xi\varepsilon U_m\left(\frac{J}{4\varepsilon}\right)\cos m\varphi\,, & m=2l\\ \Delta_m J+4\xi\varepsilon U_0\left(\frac{J}{4\varepsilon}\right), & m=2l+1\end{cases} \qquad (43)$$

Here $\Delta_m=\nu-n/m$; $\varphi=\psi-(n/m)\theta$;

$$\xi=\frac{Nr_0}{2\pi\gamma c\varepsilon} \qquad (44)$$

is the beam-beam parameter for horizontal oscillations;

$$U_0(x)=\int_0^x dt\,\frac{1-e^{-t}}{t}=\ln x+C-Ei(-x)=x\sum_{k=0}^{\infty}\frac{(-x)^k}{(k+1)^2\,k!}\,, \qquad (45)$$

$Ei(x)$ is the exponential-integral function [12], $C\simeq0.577$ is the

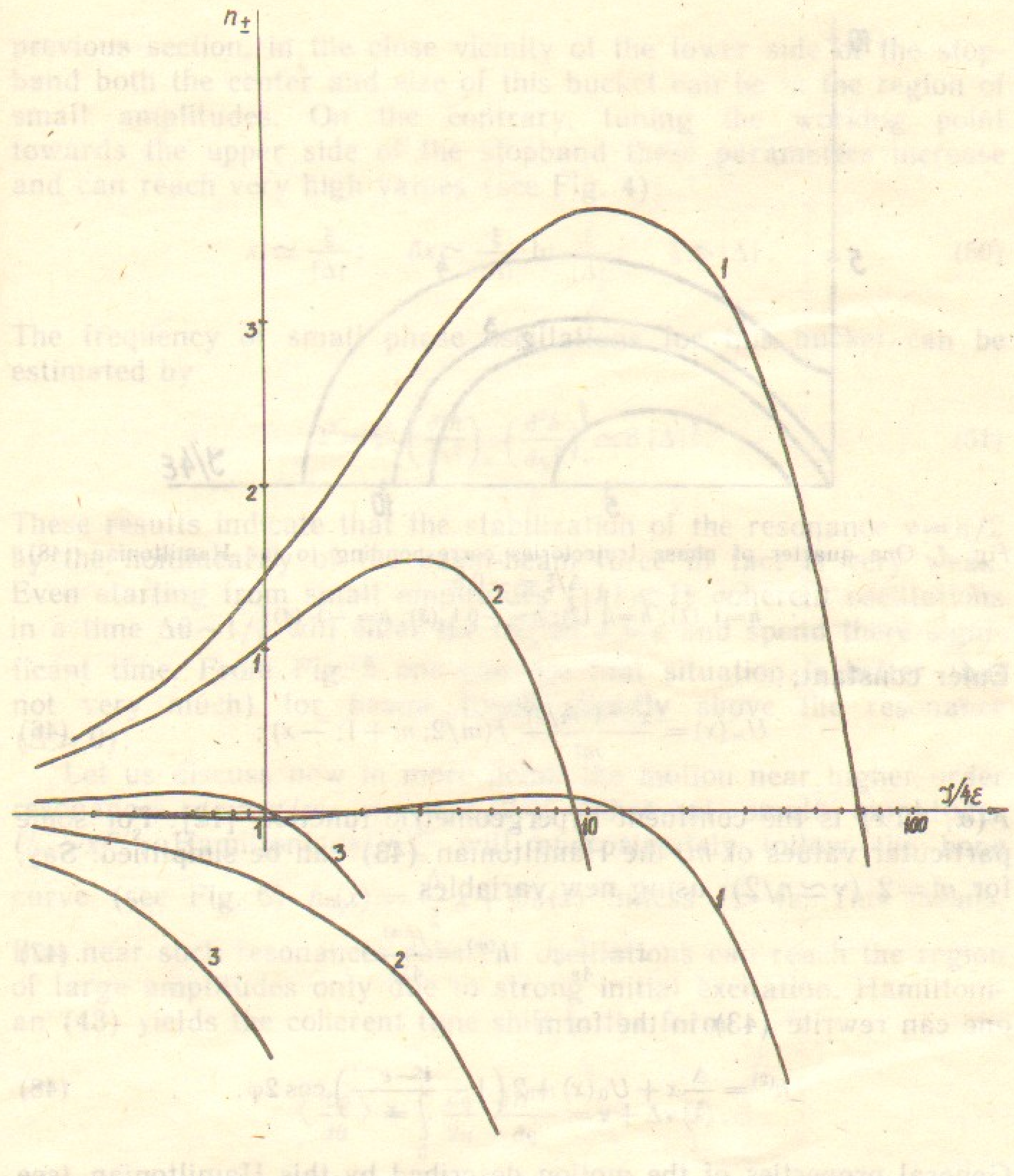

Fig. 3. Hamiltonians $H_\pm$ for resonances $\nu=n/2$: $\Delta/\xi=-0.1$ (*1*); $\Delta/\xi=-0.5$ (*2*); $\delta/\xi=-1.5$ (*3*).

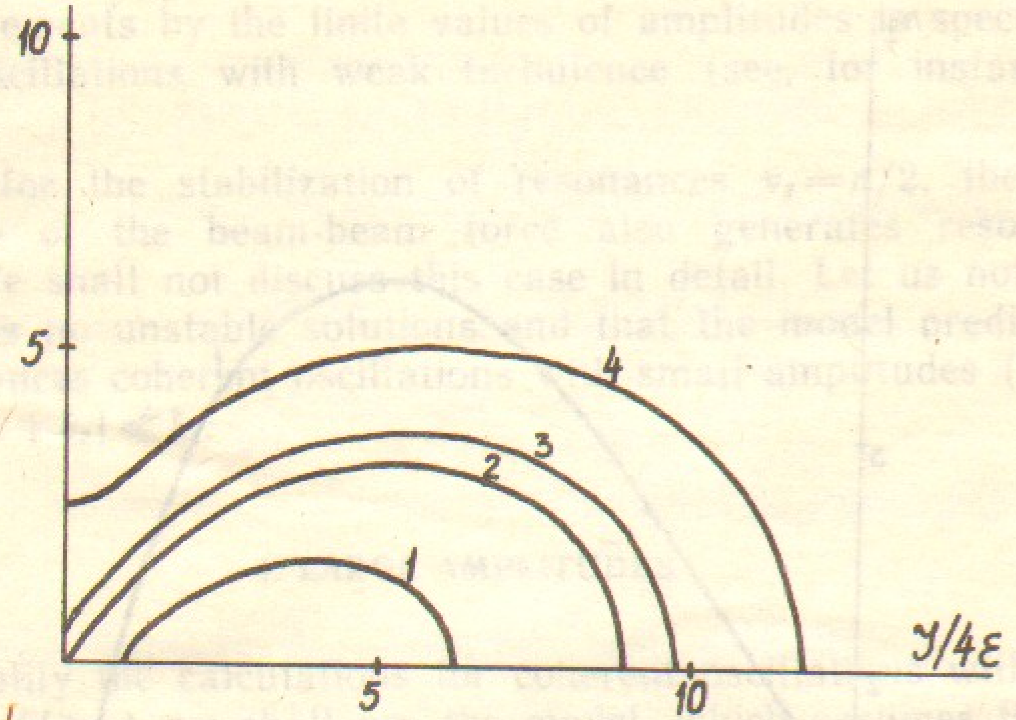

Fig. 4. One quarter of phase trajectories corresponding to the Hamiltonian (48); $\Delta/\xi = -0.5$:
$h=1$ (*1*); $h=0$ (*2*); $h=-0.1$ (*3*); $h=-1$ (*4*).

Euler constant;

$$U_m(x) = \frac{x^{m/2}\,\Gamma(m/2)}{m!}\,F(m/2;\,m+1;\,-x)\,; \tag{46}$$

$F(\alpha, \beta, x)$ is the confluent hypergeometric function [12]. For some particular values of m, the Hamiltonian (43) can be simplified. Say, for $m=2$ $(\nu \simeq n/2)$, using new variables

$$x = \frac{J}{4\varepsilon}\,; \qquad h^{(m)} = \frac{H^{(m)}}{4\xi\varepsilon} \tag{47}$$

one can rewrite (43) in the form

$$h^{(2)} = \frac{\Delta}{\xi}x + U_0(x) + 2\left(1 - \frac{1-e^{-x}}{x}\right)\cos 2\varphi\,. \tag{48}$$

General properties of the motion described by this Hamiltonian (see Fig. 3) were discussed in the previous section except only that now the amplitudes of coherent oscillations can be large $(J \gg \varepsilon)$. Since

$$\Delta\nu = \xi\frac{\partial h^{(2)}}{\partial x} = \xi\frac{1-e^{-x}}{x}, \tag{49}$$

$h^{(2)}$ has the bucket $(h=0)$ provided the working point is tuned inside the stopband (27) $(-2\xi < \Delta < 0)$. As it was found in the

previous section, in the close vicinity of the lower side of the stopband both the center and size of this bucket can be in the region of small amplitudes. On the contrary, tuning the working point towards the upper side of the stopband these parameters increase and can reach very high values (see Fig. 4):

$$x_s \simeq \frac{\xi}{|\Delta|}\,; \qquad \delta x \simeq \frac{\xi}{|\Delta|}\ln\frac{\xi}{|\Delta|}\,; \qquad \xi \gg |\Delta|\,. \tag{50}$$

The frequency of small phase oscillations for this bucket can be estimated by

$$\Omega^2 = \xi^2\left(\frac{\partial^2 h}{\partial x^2}\right)_s\left(\frac{\partial^2 h}{\partial \varphi^2}\right)_s \simeq 8\,|\Delta|^2. \tag{51}$$

These results indicate that the stabilization of the resonance $\nu = n/2$ by the nonlinearity of the beam-beam force in fact is very weak. Even starting from small amplitudes $(|h| \ll 1)$ coherent oscillations in a time $\Delta\theta \sim 1/\xi$ will enter the region $J > \varepsilon$ and spend there significant time. From Fig. 5 one can see that situation is better (but not very much) for beams tuned slightly above the resonance $(\Delta > 0)$.

Let us discuss now in more detail the motion near higher order resonances $(\nu \simeq n/m,\ m=4, \ldots, 2l)$. Since at small amplitudes $U_m \sim x^{m/2}$, Hamiltonians $h_\pm^{(m)}$ will approximately follow the bone curve (see Fig. 6) $h_0(x) = \frac{\Delta}{\xi}x + U_0(x)$ unless $J > 4\varepsilon$. This means, that near such resonances coherent oscillations can reach the region of large amplitudes only due to strong initial excitation. Hamiltonian (43) yields the coherent tune shift in the form

$$\left\langle \frac{d\psi}{d\theta} \right\rangle = \int_0^{2\pi} \frac{d\varphi}{2\pi}\,\frac{\partial H^{(m)}}{\partial\varphi} = \nu + \Delta\nu(J)\,,$$

$$\Delta\nu(J) = \xi\,\frac{1-e^{-x}}{x}. \tag{52}$$

Once $\Delta\nu(x)$ is positive the true resonant condition $\langle d\varphi/d\theta \rangle = 0$ can be valid only in the region

$$-\xi < \Delta_m < 0\,, \tag{53}$$

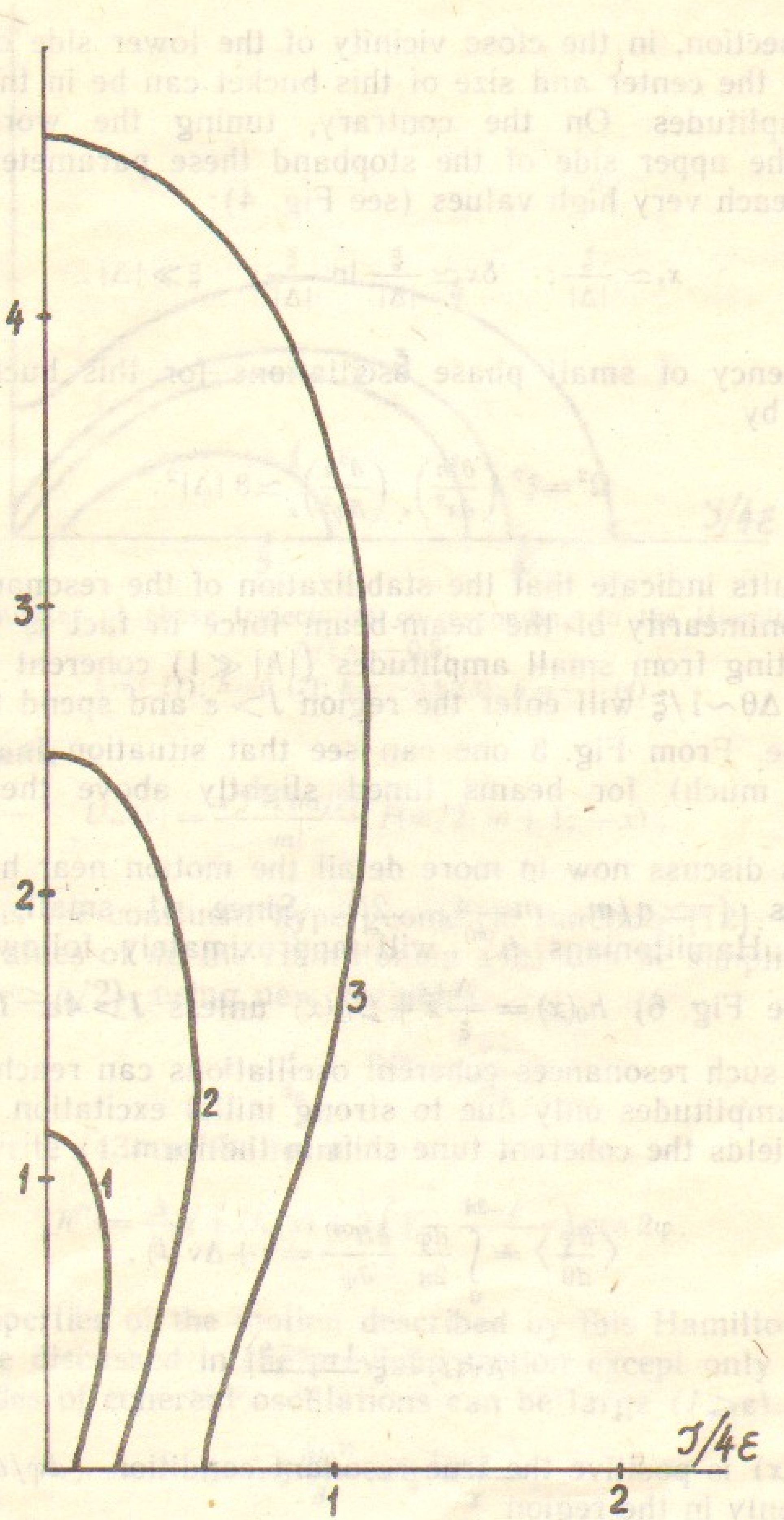

Fig. 5. The same for $\Delta/\xi=0.1$:
$h=0.1$ (1); $h=0.5$ (2); $h=1$ (3).

when the equation

$$\Delta\nu(x)=-\Delta_m \tag{54}$$

determines the amplitude of synchronous oscillations x_s. Provided $\xi\gg|\Delta_m|$, it occurs in the region of large amplitudes, where

$$\Delta\nu(x_s)\simeq\frac{\xi}{x_s}=|\Delta_m|\,,\qquad x_s=\frac{\xi}{|\Delta_m|}\,;$$

$$U_m\simeq\frac{\Gamma(m/2)}{\Gamma(m/2+1)}\,[1+O(1/x)]\simeq\frac{2}{m} \tag{55}$$

and therefore (43) can be rewritten in the form ($m=2l$)

$$h^{(m)}\simeq\frac{\Delta}{\xi}\,x+\ln x+\frac{(-1)^l\,2}{l}\cos 2l\varphi\,. \tag{56}$$

Typical behaviour of Hamiltonians $h_{\pm}^{(m)}$ is shown in Fig. 6. It indicates the existence of buckets, which are centered around x_s and corresponding values φ_s, if

$$h^{(m)}\geqslant h_c=\ln\frac{\xi}{|\Delta_m|}-1-\frac{4}{m}\,.$$

As can be seen from Fig. 6 the sizes of these buckets in x are large enough. This means that the excitation of coherent oscillation with even modest amplitudes (say, $2\div 3\sigma$) can transport particles into the region of large amplitudes.

The frequency interval, which is occupied by the bucket, is determined by the frequency of small phase oscillations

$$\Omega_m=2\,|\Delta_m|\,\sqrt{m} \tag{57}$$

via

$$\delta\nu_m=\frac{\Omega_m}{m}=\frac{2\,|\Delta_m|}{\sqrt{m}} \tag{58}$$

and generally cannot be wide in the region $\xi\gg|\Delta_m|$.

Let us now briefly discuss the action of damping on the development of coherent oscillations. Such a damping can be caused, for instance, either by cooling or by frequency spreads in the beams. For cooled beams coherent oscillations in π-mode are described by the following equations:

$$x'=-\lambda x-\xi\,\frac{\partial h^{(m)}}{\partial\varphi}\,,\qquad \varphi'=\xi\,\frac{\partial h^{(m)}}{\partial x}\,,\qquad \nu\simeq\frac{n}{m}\,, \tag{59}$$

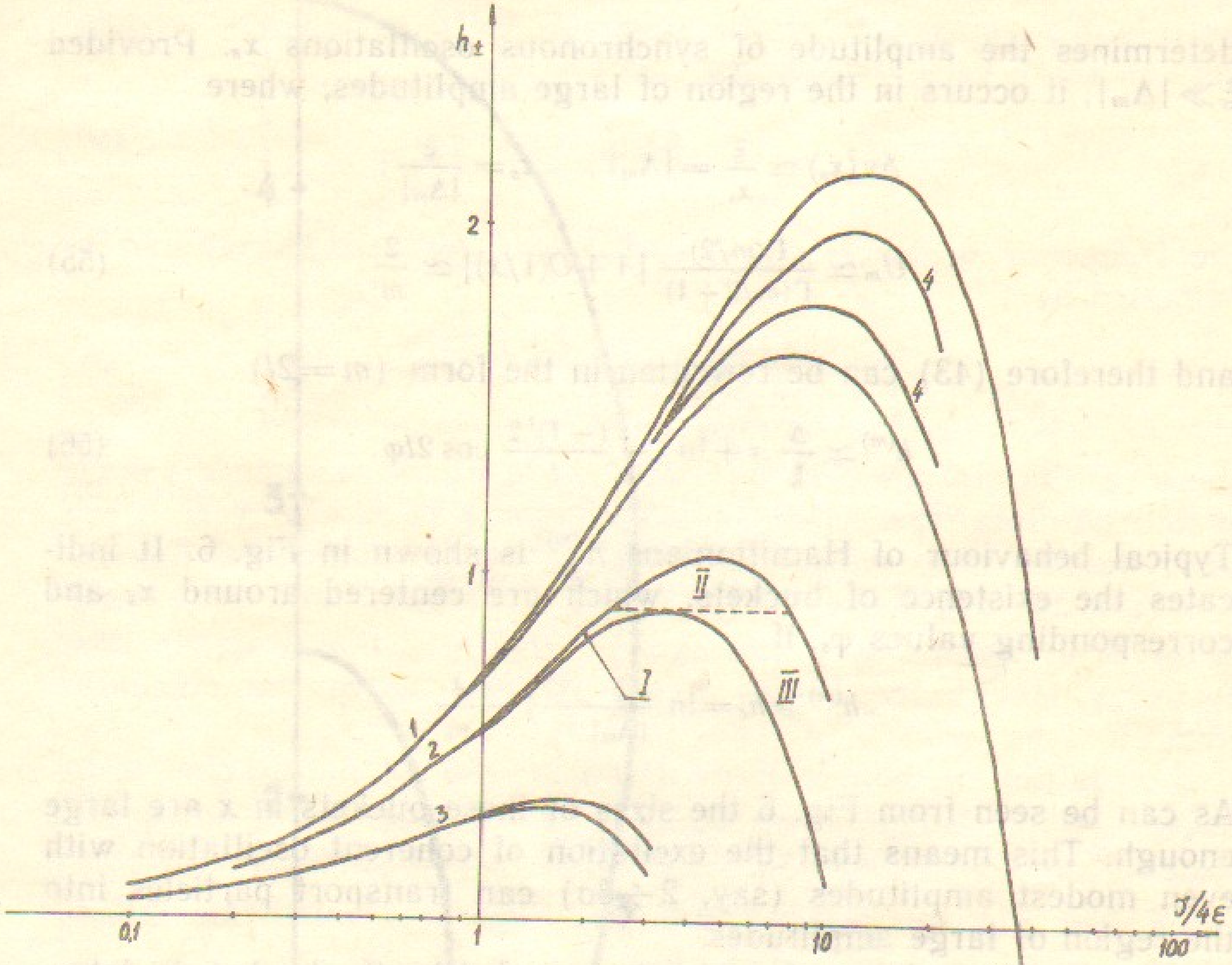

Fig. 6. Hamiltonians $H_\pm$ for resonances $\nu \simeq n/6$:
$\Delta/\xi = -0.1$ (*1*); $\Delta/\xi = -0.25$ (*2*); $\delta/\xi = -0.5$ (*3*); $\nu \simeq n/8$, $\Delta/\xi = -0.1$ (*4*).

where λ is the cooling decrement divided by the revolution frequency ω_s. Since in the region $x \gg 1$, $\partial h^{(m)}/\partial\varphi \sim 4$ and typically $\xi \gg \lambda$, the cooling will not destroy buckets outside the beam. Under these conditions the direction of relaxation can be found in the following way [14]. Using

$$\frac{dh^{(m)}}{d\theta} = \lambda x \frac{\partial h^{(m)}}{\partial x}$$

one can get that in a period of phase oscillations Θ

$$\Delta h^{(m)} = -\lambda \int_0^{\Theta} \delta\theta x \frac{\partial h^{(m)}}{\partial x} = -\lambda \oint d\varphi \, x(\varphi). \tag{60}$$

For working points above resonances $(\Delta_m > 0)$ the function $\partial h^{(m)}/\partial x > 0$ along the phase trajectory. Therefore, here $\Delta h^m < 0$ and cooling will damp coherent oscillations towards the origin $(h^{(m)}, x \to 0)$.

For working points placed below resonances $(\Delta_m < 0)$ the sigh of $\Delta h^{(m)}$ will be different in different regions, which are marked in Fig. 6. In the region I and III for the same reasons we obviously have respectively $\Delta h^{(m)} < 0$ and $\Delta h^{(m)} > 0$. Hence coherent oscillations from region I will be damped towards the origin and from region III—towards the bucket. The direction of relaxation inside the bucket (region II, see Fig. 6) depends on the ratio between λ and Ω_m. If $\lambda \gg \Omega_m$ the cooling destroys the bucket and damps oscillations towards the origin. On the contrary, if the cooling is not so strong:

$$\lambda \ll \Omega_m = 2\,|\Delta_m|\,\sqrt{m} \tag{61}$$

the oscillations will be damped towards the bottom of the bucket $(x \to x_s)$.

Within the framework of the rigid bunch model the influence of frequency spreads in the beams cannot be calculated in any straightforward way. Nevertheless, for rough estimation one may assume that it can be described by the replacement in eqs. (59) λ by the frequency spread $\delta\nu$. In the region of large amplitudes

$$\Delta\nu^{inc} \simeq \frac{2\xi\varepsilon}{J},$$

$$\delta\nu \simeq \left.\frac{\partial \Delta\nu^{inc}}{\partial J}\right|_{J=J_s} \varepsilon = \frac{2\xi\varepsilon^2}{J_s^2} = \frac{\xi}{8x_s^2} = \frac{\Delta_m^2}{8\xi},$$

$$\delta\nu \ll \xi, \Omega_m$$

this will give weak damping from the region III (see Fig. 6) towards x_s, if $\Delta_m < 0$.

5. DISCUSSION

Let us summarize some results of the paper. The calculations presented indicate that within the framework of the rigid bunch model coherent oscillations of strong-strong colliding bunches can be described using results of weak-strong theory, or multiparticle tracking with simple scale transformation. This can be done for both linear and nonlinear coherent oscillations. Nonlinear coherent oscillations of a couple of colliding beams can be captured into one of the buckets but if the number of bunches is more than 2 another buckets corresponding to the symmetry of the resonance can be also occupied. In the last case dipole oscillations can excite multipole and generally unstable coherent oscillations.

The rigid bunch model itself is adequate to the behaviour of real beams only in special conditions. Even provided the frequency spreads in beams are small enough, coherent oscillations can bring very strong modulations into single particle motion, disturbing the distributions of particles in beams [14], which breaks initial assumptions of the model. Practically this means that the model can work only for limited time intervals.

The excitation of coherent oscillations decreases the luminosity of the collider. For separated round Gaussian beams the luminosity is

$$\mathscr{L} = \mathscr{L}_0 \exp\left(-\frac{b_x^2 + b_z^2}{2\sigma^2}\right),$$

$$\mathscr{L}_0 = f_0 \frac{N^2}{4\pi\sigma^2}. \qquad (62)$$

Its value, averaged over periods of oscillations can be estimated by

$$\mathscr{L} = \mathscr{L}_0 G_x G_z,$$

$$G_\alpha = \exp\left(-\frac{J_\alpha}{8\varepsilon}\right) I_0\left(\frac{J_\alpha}{8\varepsilon}\right), \quad \alpha = x, z, \qquad (63)$$

$I_0(x)$ is the modified Bessel function [12]. Without damping of coherent oscillations suppressing factors G_α in eq. (63) depend on initial amplitudes of coherent oscillations only.

This situation changes if oscillations are damped by any mechanism (say, cooling, frequency spreads, etc.). If beams are tuned above resonances, the oscillations can decay due to damping and after this the luminosity will reach $\mathscr{L}_0$.

For beams tuned below resonances the results depend on the resonance, the strength of the kick and the ratio between damping time and the period of phase oscillations. If damping is faster than phase oscillations in the bucket, coherent oscillations will decay and the luminosity will tend to $\mathscr{L}_0$. On the contrary, if damping is slow this will take place only for small initial kicks unless the beams are tuned below $\nu = n/2$. Outside this region, as it was mentioned in the previous section, coherent oscillations will relax towards J_s. If $\xi \gg |\Delta_m|$ and therefore $J_s/4\varepsilon \simeq \xi/|\Delta_m|$, this yields:

$$\exp\left(-\frac{x_s}{2}\right) I_0\left(\frac{x_s}{2}\right) \simeq \frac{1}{\sqrt{\pi x_s}}$$

and so, for flat beams $\xi_z \gg \xi_x$:

$$\mathscr{L} \to \mathscr{L}_0 \sqrt{\frac{|\Delta_m|}{\pi \xi_z}}, \qquad (64)$$

whereas for round beams and two-dimensional oscillations:

$$\mathscr{L} \to \mathscr{L}_0 \frac{|\Delta_m|}{\pi \xi}. \qquad (65)$$

Note that, if such regimes are realized, the saturation of the luminosity will not be accompanied by the increase of beam sizes. From this point of view the resonance $\nu \simeq n/2$ remains the most dangerous.

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