32

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A b s t r a c t

The Maxwell equations in a weak gravitational field are reduced to a single scalar equation for a single unknown function. The same result is obtained also for slowly varying gravitational field of arbitrary strength up to the terms of the second order in the photon wave length. The refractive index, phase and group velocities of electromagnetic waves are calculated.

1. The Maxwell equations in a gravitational field can be written in the form

$$g^{\mu\lambda}\partial_\mu F_{\lambda\nu} + L^\lambda F_{\lambda\nu} + L^{\mu\lambda}_\nu F_{\mu\lambda} = 0 \qquad (1)$$

where $g^{\mu\lambda}$ is the metric tensor, $F_{\lambda\nu} = \partial_\lambda A_\nu - \partial_\nu A_\lambda$ is the electromagnetic field strength, $\partial_\lambda = \partial/\partial x^\lambda$, $L^\lambda = (-g)^{-1/2}\partial_\mu(\sqrt{-g}\,g^{\mu\lambda})$, and $L^{\mu\lambda}_\nu = g^{\mu\varkappa} g_{\nu\rho}\partial_\varkappa g^{\rho\lambda}$.

In what follows light cone coordinates $x^0 = t - z$ and $x^3 = (t+z)/2$ will be used. The coordinate axis x^3 is chosen so that it coincides with the null geodesics along which the electromagnetic wave propagates in the limit of infinite frequency ω. The coordinate frame is further specified by the condition that along this geodesics the metric tensor coincides with the metric tensor of the flat space-time: $g^{\mu\nu} = \eta^{\mu\nu}$, $\eta^{30} = \eta_{30} = 1$, $\eta^{mn} = \eta_{mn} = -\delta_{mn}$ $(m, n = 1, 2)$, and all the other components of $\eta^{\mu\nu}$ vanish. Moreover, the Christoffel symbols along x^3 in the chosen coordinate frame can be also made vanishing (see ref. [1]) and so the deviation of $g^{\mu\nu}$ from its flat value $f^{\mu\nu} = g^{\mu\nu} - \eta^{\mu\nu}$ is of the second order in x^α $(\alpha = 0, 1, 2)$.

Starting from equations (1) we obtain in this locally inertial frame a second order differential equation for a single unknown function. We proceed along the same lines as it is done in the case of electrodynamics in flat space-time. Differentiate eq. (1) for $\nu = n = 1,2$ with respect to x^0, take into account the second pair of the Maxwell equations

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \qquad (2)$$

and then once more use eq. (1), but with $\nu = 0$. In this way we get

$$(-2\partial_0\partial_3 + \partial_m\partial_m)F_{n0} - \partial_n\left[(f^{\mu\nu}\partial_\mu + L^\nu)F_{\nu 0} + L^{\nu\rho}_0 F_{\nu\rho}\right] + \partial_0\left[(f^{\mu\nu}\partial_\mu + L^\nu)F_{\nu n} + L^{\nu\rho}_n F_{\nu\rho}\right] = 0. \qquad (3)$$

Here m and n equal 1,2 and $\partial_m\partial_m = \partial_1^2 + \partial_2^2$.

Assume the wave length ω^{-1} to be small as compared with the scale D at which the gravitational field varies and also in comparison with the size of the wave packet d, and in its turn $d \ll D$. Going beyond the geometric optics approximation, we shall take into account terms of the order of $\omega^2 d^2$, ωd, 1, $\omega^2 d^3$, ωd^2 and $\omega^2 d^4$ (evidently $|x^\alpha| < d$; α = 0,1,2). This will allow us to calculate the dependence of the focusing or defocusing of the wave on photon polarization and corrections to the phase and group velocities.

We shall look for the solution of the form

$$F_{n0} = \exp\{-i\omega x^0 + i\omega\psi + i\varphi + i\omega^{-1}\chi\} \tag{4}$$

It can be seen from eq. (3) that $\psi \sim (x^\alpha)^2$ at $x^\alpha \to 0$ (α = 0, 1,2) whereas φ and χ are in general nonvanishing.

Eqs. (2) with the account of expression (4) lead to the following estimates

$$F_{30} \sim F_{12} \sim (d + \omega^{-1}) F_{n0},$$

$$F_{3n} \sim (d^2 + \omega^{-1}) F_{n0}.$$

Hence with the taken accuracy we obtain two equations for two large components of the field:

$$\begin{aligned}
&[-2\partial_0\partial_3 + \partial_m\partial_m - f^{\mu\nu}\partial_\mu\partial_\nu - (L^0 + \partial_0 f^{00})\partial_0 - \\
&-(L^m + 2\partial_0 f^{0m})\partial_m - \partial_0 L^0] F_{n0} + \\
&+[(L_n^{m0} - L_n^{0m} - \partial_n f^{0m})\partial_0 - (L_0^{m0} - \partial_0 f^{0m})\partial_n + \\
&+(L_n^{30} - \partial_n f^{03})\partial_m + (L_n^{mk} - L_n^{km} - \partial_n f^{km})\partial_k - \\
&-\partial_n L^m - \partial_n L_0^{m0} + \partial_0 L_n^{m0}] F_{m0} = 0
\end{aligned} \tag{5}$$

where m, n, k = 1,2.

It is convenient to introduce new field variables defined by

$$F_{n0} = E_n + \frac{1}{2} f^{nk} E_k. \tag{6}$$

The system of equations for E_n, obtained with our accuracy, has an antisymmetric nondiagonal part. It can be diagonalized in terms of helicity states

$$E_\lambda = \frac{1}{\sqrt{2}}(-\lambda E_1 - i E_2), \quad \lambda = \pm 1. \tag{7}$$

The final equation is

$$\begin{aligned}
&\{-2\partial_0\partial_3 + \partial_m\partial_m - f^{00}\partial_0^2 - 2f^{03}\partial_0\partial_3 - f^{km}\partial_k\partial_m + \\
&+2f_{3k}\partial_0\partial_k + (2\partial_0 g_{33} - \partial_m g_{m3} + \tfrac{1}{2}\partial_3 g_{mm})\partial_0 - \\
&-(3\partial_0 g_{3m} - \partial_k g_{km} - \partial_m g_{03} + \tfrac{1}{2}\partial_m g_{kk})\partial_m + \\
&+\partial_0^2 g_{33} - \partial_0\partial_m g_{m3} + R_{1212} + i\lambda[(\partial_1 g_{23} - \partial_2 g_{13})\partial_0 + \\
&+(\partial_0 g_{13} - \partial_1 g_{30} + \partial_2 g_{12} - \partial_1 g_{22})\partial_2 - \\
&-(\partial_0 g_{23} - \partial_2 g_{30} + \partial_1 g_{12} - \partial_2 g_{11})\partial_1 - \\
&-\tfrac{1}{2}\partial_m(\partial_1 g_{2m} - \partial_2 g_{1m}) + 2R_{1230}]\} E_\lambda = 0.
\end{aligned} \tag{8}$$

Here as usually summation goes in all repeating indices, $g_{\mu\nu} = \eta_{\mu\nu} - f_{\mu\nu}$, $f^{00} = f_{33} + 2f_{03}f_{33} - f_{3k}f_{3k}$. The curvature tensor in our approximation is equal to

$$R_{\mu\nu\varkappa\lambda} = \frac{1}{2}(\partial_\mu\partial_\lambda g_{\nu\varkappa} + \partial_\nu\partial_\varkappa g_{\mu\lambda} - \partial_\mu\partial_\varkappa g_{\nu\lambda} - \partial_\nu\partial_\lambda g_{\mu\varkappa})$$

although the gravitational field is not assumed to be weak.

2. In the weak field approximation eq. (8) considerably simplifies because the derivatives $\partial_3 E_\lambda$ and $\partial_m E_\lambda$ are proportional to the gravitational field. Thus we get

$$(-2\partial_0\partial_3 + \partial_m\partial_m - \omega^2 g_{33} - i\omega V + W)E_\lambda = 0 \tag{9}$$

where

$$V = 2\partial_0 g_{33} - \partial_m g_{m3} + \frac{1}{2}\partial_3 g_{mm} + i\lambda(\partial_1 g_{23} - \partial_2 g_{13}),$$

$$W = R_{1212} + \partial_0^2 g_{33} - \partial_0\partial_m g_{m3} + \frac{1}{2}\partial_0\partial_3 g_{mm} +$$

$$+ i\lambda\left[2R_{1230} - \frac{1}{2}\partial_m(\partial_1 g_{2m} - \partial_2 g_{1m})\right].$$

It is noteworthy that this equation is valid without the assumption of smallness of the electromagnetic wave length. Just for this purpose we retain the term $\frac{1}{2}\partial_0\partial_3 g_{mm}$ in W which was omitted in eq. (8).

We look for the solution of eq. (9) as (4) despite eq. (9) being valid in the weak field approximation only and so the exponentiation of small term ($\omega\psi + \varphi + \omega^{-1}\chi$) is not absolutely consistent. This form however is convenient for calculation of the phase and group velocities of the wave.

Substituting expression (4) into eq. (9), we obtain the following coupled equations defining ψ, φ and χ in the lowest order in gravitational field:

$$2\partial_3\psi = -g_{33},$$

$$2\partial_3\varphi = i(-2\partial_0\partial_3 + \partial_m\partial_m)\psi - iV, \tag{10}$$

$$2\partial_3\chi = i(-2\partial_0\partial_3 + \partial_m\partial_m)\varphi + W.$$

In particular it follows from (10)

$$\partial_3\chi = \frac{1}{2}(W - \partial_0 V + \partial_0^2 g_{33}) +$$

$$+ \frac{1}{4}\int^{x^3} dx^{3'}\partial_m\partial_m(V - 2\partial_0 g_{33}) + \tag{11}$$

$$+ \frac{1}{8}\int^{x^3} dx^{3'}\int^{x^{3'}} dx^{3''}\partial_k\partial_k\partial_m\partial_m g_{33}.$$

Then with the help of the identities

$$\partial_m\partial_m g_{33} = 2R_{33} - \partial_3^2 g_{mm} + 2\partial_3\partial_m g_{m3},$$

$$\partial_m\partial_m(\partial_1 g_{23} - \partial_2 g_{13}) = \partial_3\partial_m(\partial_1 g_{2m} - \partial_2 g_{1m}) +$$

$$+ 2(\partial_1 R_{23} - \partial_2 R_{13} + \partial_3 R_{1230})$$

expression (11) can be rewritten as

$$\partial_3\chi = \frac{1}{2}(R_{1212} + i\lambda R_{1230}) +$$

$$+ \frac{i\lambda}{2}\int^{x^3} dx^{3'}(\partial_1 R_{23} - \partial_2 R_{13}) + \frac{1}{4}\int^{x^3} dx^{3'}\int^{x^{3'}} dx^{3''}\partial_m\partial_m R_{33}. \tag{12}$$

Note that $\partial_3\chi$ does not depend explicitely neither on the metric tensor, nor on the Christoffel symbols and is determined by the curvature tensor only. In the source-free space region where $R_{\mu\nu} = 0$, integrands in eq. (12) vanish and the result has the remarkably simple form

$$\partial_3\chi = \frac{1}{2}(R_{1212} + i\lambda R_{1230}). \tag{13}$$

3. Now we derive the expression for the phase velocity of a wave in a gravitational field relative to an observer at rest. Three-velocity of a point which is displaced by four--vector dx^μ is [2] :

$$u^i = \frac{\sqrt{g_{tt}}\,dx^i}{g_{tt}\,dx^t + g_{tj}\,dx^j}. \tag{14}$$

Here indices i and j run over space values x, y and z, and t is the time coordinate.

Let a surface of a constant wave phase be described by the equation $\Phi(x,y,z,t)=0$. The displacement of a point on this surface can be presented in the form

$$dx^\mu \sim p^\mu + \alpha v^\mu . \tag{15}$$

where $p_\mu = \partial_\mu \Phi$ and $v^\mu = (g_{tt})^{-1/2}(1,0,0,0)$ is the four-velocity of an observer. The condition of a constant phase $d\Phi = 0$ means that $p_\mu dx^\mu = 0$. Hence

$$dx^\mu \sim p^\mu - \frac{p_\nu p^\nu}{p_\lambda v^\lambda} v^\mu . \tag{16}$$

Substituting expression (16) into (14), we find [3]

$$u^{-2} = 1 - \frac{g^{\mu\nu} p_\mu p_\nu}{p_t^2} g_{tt} \tag{17}$$

where $u^2 = (-g_{ij} + g_{ti} g_{tj}/g_{tt}) u^i u^j$ is the three-velocity squared.

The inverse three-velocity can be interpreted as the refractive index n for light in a gravitational field. After simple calculations we get

$$u = n^{-1} = 1 - Re\left(\frac{1}{\omega}\partial_3 \varphi + \frac{1}{\omega^2}\partial_3 \chi\right). \tag{18}$$

One can see from eqs. (10) and (13) that on the axis x^3 the following result is valid

$$u = n^{-1} = 1 - \frac{1}{2\omega^2} R_{1212} . \tag{19}$$

It follows, however, from the same formulas (10) and (13) if the explicit expression for V is taken into account that in arbitrary coordinates the refractive index depends not only on the curvature tensor, and it is not therefore a covariant quantity. It is quite natural: e.g., in a coordinate frame where the axes x^1, x^2 rotate along the trajectory, n should evidently depend on circular polarization λ .

Using eq. (18) one can calculate group velocity of the wave packet:

$$v = \left[\frac{d}{d\omega}(\omega n)\right]^{-1} = 1 + \frac{1}{\omega^2} Re\, \partial_3 \chi = 1 + \frac{1}{2\omega^2} R_{1212} . \tag{20}$$

So the first corrections to the refractive index (in the coordinates used!) and to the group velocity are determined by the scalar curvature of the two dimensional surface orthogonal to the wave packet trajectory.

The sign of the correction is not definite so the group velocity can be both larger and smaller than unity. This however does not contradict the causality condition due to which the wave front velocity should be bounded by unity. This velocity however is equal to [4] $\lim_{\omega\to\infty} n^{-1}(\omega)$ and in accordance with eq. (18) is of course unity. In this situation $v > 1$ means only a corresponding deformation of wave packet.

The wave phase Φ has a nonzero imaginary part which leads to nonzero imaginary part of the refractive index:

$$\begin{aligned} Im\, n &= Im\left(\frac{1}{\omega}\partial_3 \varphi + \frac{1}{\omega^2}\partial_3 \chi\right) = \\ &= -\frac{1}{2\omega}\left(\partial_0 g_{33} + \int^{x^3} dx^{3\prime} R_{33}\right) + \\ &+ \frac{\lambda}{2\omega^2}\left[R_{1230} + \int^{x^3} dx^{3\prime}\left(\partial_1 R_{23} - \partial_2 R_{13}\right)\right]. \end{aligned} \tag{21}$$

On the axis x^3 $\partial_0 g_{33} = 0$ and if in addition the wave propagates in an empty space (where $R_{\mu\nu} = 0$), the expression for $Im\, n$ simplifies considerably:

$$Im\, n = \frac{\lambda}{2\omega^2} R_{1230} . \tag{22}$$

So if $R_{1230} \neq 0$ then the wave amplitude rises for one sign of the circular polarization and decreases for another.

4. In connection with the results obtained above we would like to comment on the existing in the literature statement about a superluminal light propagation in a gravitatio-

nal field when radiative corrections are taken into account. In paper [5] it was noted that QED radiative corrections lead to a change of the characteristics of the photon wave equations for frequencies that are small in comparison with the electron mass m_e. The contribution to the refractive index which results from this effect proves to be negative for one of the photon polarization states, and does not depend on the frequency in the above mentioned limit $\omega \ll m_e$. Assuming that $Im\, n$ is positively definite and using the dispersion relation for $n(\omega)$ the authors of Ref. [5] conclude that $n(\omega \to \infty) < n(\omega = 0)$ and thus $n(\omega \to \infty) < 1$. Consequently for one of the polarization states causality is explicitely violated. The same statement is made in Ref. [6] for neutrino.

From the above results two shortcomings of this consideration can be seen. First, the value of $n(0)$ in a gravitational field in fact was not calculated. In particular there exists the correction (19) to the refractive index which is much larger than that found in paper [5]. Expression (19) however is valid only for $\omega D \gg 1$ (D is the characteristic scale of the gravitational field), and so it can-not be directly used for $\omega = 0$. Second, the sign of $Im\, n$ is not definite for wave propagation in nonhomogeneous media as is seen from eqs. (21) and (22). The physical reason of it is evident. In a homogeneous stable (i.e., without particle production) medium the amplitude of a wave changes only because the particles go out of the beam. This corresponds to the condition $Im\, n > 0$. If the medium is not homogeneous, then the focusing (or bunching) of the beam is possible. This leads to an increase of the wave amplitude and thus corresponds to $Im\, n < 0$.

This statement can be illustrated by the following one-dimensional quantum mechanical example. The quasiclassic expression for the wave propagating along x axis in a potential $U(x)$ is

$$\psi = \left[\frac{\omega^2}{\omega^2 - U(x)}\right]^{1/4} exp\left\{ i \int^x dx' \sqrt{\omega^2 - U(x')} \right\}$$

For $\omega \gg U$ it can be rewritten as

$$\psi = exp\left\{ i\omega x - \frac{i}{2\omega}\int^x dx' U(x') + \frac{1}{4\omega^2} U(x) \right\} \tag{23}$$

Hence $Im\, n(\omega; x) = -\frac{1}{4\omega^2}\frac{dU}{dx}$. Evidently the sign of $Im\, n$ is not definite.

Thus the conclusion made in Refs. [5,6] about the causality violation does not seem justified. Unfortunately our early attempt [7] to refute the conclusion of these papers also was not successful.

6. Return to an arbitrary gravitational field in which wave propagation is described by eq. (8). Making substitution (4) one can easily get coupled nonlinear equations for the functions ψ, φ and χ. The size of the wave packet being small in comparison with the characteristic scale of the gravitational field, it is convenient to expand these functions in powers of x^α (α = 0,1,2). In particular, the accuracy with which we work, allows us to keep in ψ terms up to the fourth order:

$$\psi = \frac{1}{2}\psi_{\alpha\beta}(x^3)x^\alpha x^\beta + \frac{1}{6}\psi_{\alpha\beta\gamma}(x^3)x^\alpha x^\beta x^\gamma + \frac{1}{24}\psi_{\alpha\beta\gamma\delta}(x^3)x^\alpha x^\beta x^\gamma x^\delta. \tag{24}$$

From the equation for ψ

$$2\partial_3\psi - 2\partial_0\psi\,\partial_3\psi + \partial_n\psi\,\partial_n\psi + 2f^{\mu 0}\partial_\mu\psi - f^{\mu\nu}\partial_\mu\psi\,\partial_\nu\psi = f^{00} \tag{25}$$

the following equations for $\psi_{\alpha\beta}$ can be obtained

$$\partial_3\psi_{\alpha\beta} + \psi_{\alpha n}\psi_{\beta n} = R_{3\alpha 3\beta} \tag{26}$$

(here α, β = 0,1,2 and n = 1,2).

The functions ψ_{11}, ψ_{12}, ψ_{22} satisfy an independent subsystem of nonlinear equations

$$\partial_3\psi_{km} + \psi_{kn}\psi_{mn} = R_{3k3m}. \tag{27}$$

After ψ_{mn} are found, the calculation of ψ_{01}, ψ_{02} is reduced to the solution of a system of linear equations

$$\partial_3 \psi_{0m} + \psi_{0n}\psi_{mn} = R_{303m}. \tag{28}$$

And finally, at known ψ_{0n} the function ψ_{00} is found in quadratures:

$$\psi_{00} = \int^{x^3} dx^{3'} \{ R_{3030} - \psi_{0n}\psi_{0n} \}. \tag{29}$$

The equations for the functions $\psi_{\alpha\beta\gamma}$ and $\psi_{\alpha\beta\gamma\delta}$ can be also easily obtained from eq. (25). We do not write them down because they are rather cumbersome.

The equations for φ and χ prove to be linear. In our approximation φ can be written as

$$\varphi = \varphi^{(0)}(x^3) + \varphi_\alpha(x^3)x^\alpha + \frac{1}{2}\varphi_{\alpha\beta}(x^3)x^\alpha x^\beta \tag{30}$$

and χ should be considered for $x^\alpha = 0$ only. The equation for $\varphi^{(0)}(x^3)$ is simple:

$$\partial_3 \varphi^{(0)} = \frac{i}{2}(\psi_{11} + \psi_{22}), \tag{31}$$

but other equations, their derivation being also straightforward, take too much space and so we do not write them down.

Using eq. (17), one can find refractive index n at $x^\alpha = 0$:

$$n = 1 + \frac{1}{\omega^2}\left[Re\, \partial_3 \chi + \frac{1}{2}(Re\, \varphi_1)^2 + \frac{1}{2}(Re\, \varphi_2)^2 \right] \tag{32}$$

where φ_n are defined by the expression (30). It follows in particular from (32) that the refractive index, and phase and group velocities as well, are independent of the photon polarization in our locally inertial coordinate frame.

One can also easily find by iterations nonlinear corrections to the real and imaginary parts of the refractive index.

For a further general investigation of the solution one should know the explicit expression for the gravitational field.

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О РАСПРОСТРАНЕНИИ ЭЛЕКТРОМАГНИТНЫХ ВОЛН В ГРАВИТАЦИОННОМ ПОЛЕ

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