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# THE NATURE AND PROPERTIES OF THE DYNAMICAL CHAOS

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THE NATURE AND PROPERTIES OF
THE DYNAMICAL CHAOS*

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A b s t r a c t

Some recent developments in classical Hamiltonian mechanics related to the phenomenon of dynamical (deterministic) chaos are briefly discussed. Those include: the KAM integrability; peculiarities of weakly nonlinear dynamics; the algorithmic randomness; the chaos border and long-time correlations.

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## 1. Introduction

Until recently statistical properties of a physical system have been derived with the help of some special postulates, or Ansätze, such as Gibbs' microcanonical distribution in equilibrium statistical mechanics or Boltzmann's molecular chaos, Bogolyubov's correlation decay, Prigogine's causality condition, van Hove's random phase approximation and the like in nonequilibrium mechanics. Those assumptions are especially plausible and natural in the so-called thermodynamic limit, that is in the limit of infinite number of degrees of freedom (DF) for a given density. In a finite dimension, and the more so in just a few DF those conjectures are generally false. Yet, such a classical approach to the statistical mechanics can be, nevertheless, saved by introducing an extrinsic "noise" with given statistical properties (correlations) - the method which can be traced back to Langevin.

Nowadays, tremendous progress in the modern ergodic theory enables us, in principle, both to find out the conditions for as well as to derive the peculiarities of the statistical, or chaotic (stochastic), behavior on a purely dynamical basis, i.e. by the analysis of dynamical (deterministic) evolution equations free of any random element or assumption. In other words, in this new approach the chaotic behavior comes out as a particular regime of the dynamical evolution. To reveal peculiarities of this regime one needs, of course, to make use of some special statistical description and notions, such as distribution function, coarse graining, symbolic trajectories etc. The latter ones, however, should not be confused with statistical properties, or chaotic behavior, which do not depend on the description.

As far as it concerns the classical mechanics that dynamical chaos happens to be observationally undistinguishable from the "true" random process, the latter being understood as unpredictable, or, better to say, unreproducible (see Section 5). The quantum dynamics appears to be less chaotic, if any, and is certainly much less known thus far /7,8/.

In the present talk I am going to briefly discuss some of



these more or less recent developments, refering to a couple of simple models of the Yang-Mills field as examples.

A mathematical review of the topic in question can be found in Refs. /1-5/, that by a physicist is in Refs. /6-9/, and recent popular presentations are in Refs. /10-12/.

In what follows we shall restrict ourselves to classical Hamiltonian dynamics, that is to nondissipative but not necessarily conservative (in energy) dynamical systems. A typical problem to be studied - the Poincaré fundamental dynamical problem - is the influence of a weak perturbation on a completely integrable unperturbed system which, for the time being, we assume to be nonlinear, or nondegenerate. It means that determinant

$$\left|\frac{\partial \omega_i}{\partial I_k}\right| = \left|\frac{\partial^2 H_0}{\partial I_i \partial I_k}\right| \neq 0 \qquad (1.1)$$

with $\omega_i$ , $I_k$ as the frequencies and action variables, respectively, and $H_0$ the unperturbed Hamiltonian. A more difficult for analysis case of the linear (isochronous) unperturbed system will be considered in Sec. 4. Particularly, Eq. (1.1) implies one-to-one correspondence between I- and $\omega$-subspaces of the dynamical (phase) space.

## 2. KAM integrability

The concept of integrability has little to share with actual evaluation of a motion trajectory, the latter can always be done numerically, if necessary. Instead, it is a basic characteristic of the motion (phase space) structure, namely, of its decomposition into elementar (irreducible) dynamical components. In the natural at this Conference group-theoretical language the problem of integrability is also the problem of the maximal symmetry group of a dynamical system. For a closed system the minimal one is the Poincaré group implying the ten well known integrals of motion. For the sake of brevity one uses to say just about energy only, assuming the others to have been eliminated beforehand. Thus, the integrability refers to additional specific isolating integrals related to some "hidden" symmetry of the system. If there is nothing to "hide", the motion is said to be transitive, or ergodic, on the energy surface. In this case the whole phase space decomposes into one-parameter family of ergodic components, each comprising the entire energy surface. The opposite case corresponds to the full set of N (the number of DF) commuting (in involution) integrals which reduce the elementar dynamical component to a quasi-periodical motion on a torus. For almost all tori the motion is ergodic (on torus), or nonresonant. This is the classical view of the integrability problem coming back to Poincaré.

A lucid presentation of our current understanding of this problem can be given via resonance analysis. The latter is especially convenient to carry out in the $\omega$-subspace where each resonance

$$\sum_{i=1}^{N} m_i \omega_i + \sum_{k=1}^{M} n_k \Omega_k \equiv (m,\omega) + (n,\Omega) = 0 \quad (2.1)$$

is plainly a plane. Here $m_i$, $n_k$ are integers, and the external perturbation is assumed to be quasi-periodic with M basic frequencies $\Omega_k$. Note that generally $m_i$, $n_k$ are any integers, and, hence, the resonance set is everywhere dense in the phase space. Yet, its measure (volume) is zero, and this constitutes a singularity whose dynamical implications had not been properly recognized until the KAM theory was created, mainly, due to Kolmogorov, Arnold and Moser /13, 1 - 3/ (also see Ref. /18/). In particular, the famous Poincaré theorem /14/ (Sections 81-83), that a generic Hamiltonian system is nonintegrable, being formally true has been actually misleading for a long time, at least, for physicists (see, e.g., Ref. /19/). The trick is that the theorem ensures the absence of analytical integrals of motion. This mathematical subtlety is not a merely technical requirement. Instead, it is essential, indeed, since the actual distruction of the motion integrals does occur, as already Poincaré was aware of, just on the resonance set due to the well known small denominators in the perturbation series. Obviously, the function which does not exist



on an everywhere dense set cannot be analytical. But what about that? Do we really need analytical integrals of motion? Or, to put the question in other way, have we to follow Poincaré in his somewhat implicit but fairly "obvious" assumption that the motion integrals comprise, at least, some solid region including resonance surfaces? The KAM theory teaches us that we have not. Yet, one can say also that there are (at least!) two different notions of integrability:

i) the Poincaré, or global, integrability which refers to the over-all through the phase space integrals; this notion somewhat exaggerates the letter of Poincare's idea retaining, however, its spirit, and is close to the modern notion of the complete integrability;

ii) the KAM integrability restricted to a nowhere dense nonresonant set in the phase space.

Let us dwell on the latter new notion somewhat longer. First, what are the conditions for KAM integrability? They are essentially three:

i) the unperturbed globally, or Poincaré, integrable system is nonlinear, or nondegenerate (1.1);

ii) the perturbation is sufficiently smooth, i.e. it belongs to a class $C^{\ell}$ with some $\ell > \ell_{cr}$;

iii) the perturbation is sufficiently weak, i.e. the perturbation parameter $\varepsilon \lesssim \varepsilon_{cr}$.

The critical perturbation smoothness $\ell_{cr}$ (the number of continuous mixed partial derivatives with respect to both $N$ angles $\theta_i$ as well as M time variables $\varphi_k = \Omega_k t + \varphi_k^{\circ}$) can be estimated by the resonance overlap criterion as

$$\ell_{cr} \approx 2(N+M) - 2 \quad (2.2)$$

which is a generalization of the result in Ref. /6/ (Section 4.5) to an explicit quasiperiodic time dependence of the Hamiltonian. This may be compared to the rigorous upper estimate, for M = 0, due to Moser /15/:

$$\ell_{cr} \leq 2N + 2 \quad (2.3)$$

Fc mapping the quantity $\ell_{cr}$ (2.2) increases by 2.

In particular, $\ell_{cr} = 2$ if M = 0, and N = 1 /6/. The best rigorous upper bound in the latter case, again due to Moser /3/, is $\ell_{cr} \leq 4$. For $\ell \leq \ell_{cr}$ the critical perturbation strength ( $\varepsilon_{cr}$ ) does not exist, i.e. the system is not even KAM integrable at any nonzero $\varepsilon \to 0$. Takens /16/ has proved this for a particular mapping with $\ell = 2$ (M = 0; N = 1). Note that our $\ell$ characterizes the Hamiltonian (or generating function) and is bigger by one as compared to that in Refs. /3,16/.

Evaluation of the critical perturbation strength $\varepsilon_{cr}$ (for $\ell > \ell_{cr}$) is a more tricky problem. We just mention here the analytical case (for detail see Ref. /6/, Section 4.6). Let the perturbation be analytic in both $\theta_i$ and $\varphi_k$ within the strip of half-width $\sigma$. A particular question is how does $\varepsilon_{cr}$ scale with $\sigma$? An interesting point is that the powerful techniques, developed by Moser /3/ to deal with a smooth perturbation, can be applied back to analytical perturbation to get a fairly efficient estimate /6/

$$\varepsilon_{cr} \propto \sigma^{\left(\frac{3}{4}\ell_{cr} + C\right)} \tag{2.4}$$

where constant C = 3/2 from the overlap criterion, and $C \leq 5/2$ from the rigorous upper bound (2.3).

## 3. Separatrix stochastic layers and the Arnold diffusion

What is the nonintegrable set like, under the conditions of KAM integrability? The first estimate for its relative size and the total measure was of the order of $\varepsilon^{1/2}$, that is of the full width of a nonlinear resonance $(\Delta\omega)_r$. However, further studies /17,6,9/ revealed that the actual measure of the chaotic component, which forms very narrow stochastic layers about resonance separatrices, is a great deal smaller:

$$\xi = \frac{(\Delta\omega)_s}{(\Delta\omega)_r} \sim e^{-\frac{A(\omega_i)}{(\Delta\omega)_r}} \tag{3.1}$$

where $(\Delta\omega)_s$ is the width of a stochastic layer, and factor $A(\omega_i)$ depends on dynamical variables. In simple cases the



evaluation of estimate (3.1), based upon the overlap criterion, is fairly accurate (see Ref. /6/, Section 6.2, and Ref. /20/). It is worth noting that estimate (3.1) and the like have been obtained not by means of the powerful KAM techniques for the construction of convergent perturbation series but, instead, using a routine asymptotic method of averaging /21/. However, the crucial new feature of our approach is introducing a new (resonant) perturbation parameter $\xi$ (3.1) (instead of original $\varepsilon$ ) which has to be explicitly calculated /6,20/.

What are dynamical implications of the nonintegrable component of motion? They depend on topology of the phase space. In case of N = 2 (for a closed system) stochastic layers are separated from each other by integrable components, and their influence is negligible /1,2/. Namely, the motion remains stable (in action variables) for <u>all</u> initial conditions, and the averaged globably integrable system approaches the true motion with only exponentially small ($\sim\xi$) ineradicable error.

However, in many dimensions ( $N > 2$) the motion picture changes drastically since all those layers do merge and form united everywhere dense "web" over which trajectory can, and generally does, approach arbitrarily close any point on the energy surface /17/. Yet, it does not mean the ergodicity of motion since the total measure of the web is still exponentially small. For this latter reason the motion on the web, which is called the Arnold diffusion, may appear to be practically unimportant. In many cases it is true, indeed, the more so that the diffusion rate is also exponentially small /22,6,9/. Nevertheless, Arnold diffusion certainly signifies a real (and universal /6/!) motion instability contrary to the assertion in Ref. /23/. The latter confusion is caused by an artificial notion of "allowable" solution, introduced in Ref. /23/, which just excludes the unstable initial conditions on the web.

On the other hand, the whole problem is improper since the stochastic web is everywhere dense. There are several methods of so-called regularization of the problem, i.e. of its unambiguous formulation which would not depend on infinitesimal changes in initial conditions. One way is to bound the motion time interval from above by an arbitrary but finite value. It converts the everywhere dense resonance web into a finite mesh grid of "working" (sufficiently strong) resonances, and the problem acquires the physical meaning.

Another regularization method relates to introducing, or, better to say, to taking account of some always present arbitrarily weak but finite external "noise" which results in an additional diffusion /6/. It also leaves a finite resonance set where Arnold diffusion exceeds (in rate) the effect of noise. Besides, the latter brings all the trajectories to one of the "working" stochastic layers and provides Arnold diffusion for all initial conditions.

Thus, Poincaré, or local, nonintegrability does not generally imply any significant change in the motion structure. Instead, the KAM integrability takes generally place with, at worst, an exponentially slow Arnold diffusion.

Note that separatrix splitting, and, hence, formation of a stochastic layer, which sometimes is used as a criterion for nonintegrability (see, e.g., Refs. /24,25/), relates just to a local nonintegrability and does not contradict with the KAM integrability as was explained above.

A large-scale, or global, chaos sets in as a result of breaking down the KAM integrability as perturbation exceeds the critical level ($\varepsilon \gtrsim \varepsilon_{cr}$). Let us mention that generally, apart from some special cases as, for instance, the standard mapping (see Refs. /6,20/), any definite critical value of the perturbation strength does not exist for a Hamiltonian system. Instead, the measure of chaotic component is continuously increasing as $\varepsilon$ grows. In other words, the parameter $\varepsilon_{cr}$ has meaning in order of magnitude only. However, the chaos border in phase space has the definite meaning and important implications for chaotic dynamics (see Section 5 and Ref. /20/).



4. Weak nonlinearity

If unperturbed oscillations are linear (isochronous), and, hence, $|\partial\omega_i/\partial I_k| \equiv 0$ (1.1), the nonlinearity comes out from perturbation terms only. This situation is rather typical in applications. Consider, for example, one of Matinyan's model for classical spacially homogeneous Yang-Mills (YM) field in a Higgs vacuum /26/:

$$H(I_k,\theta_k) = H_0(I_k) + V(I_k,\theta_k)$$

$$H_0 = \frac{1}{2}\left(E_1^2 + E_2^2 + \omega_1^2 A_1^2 + \omega_2^2 A_2^2\right) = \omega_1 I_1 + \omega_2 I_2$$

$$V = \frac{A_1^2 A_2^2}{2} = \bar{V} + V_r + \tilde{V} \tag{4.1}$$

$$\bar{V} = \frac{I_1 I_2}{2\omega_1\omega_2}; \quad V_r = \frac{I_1 I_2}{4\omega_1\omega_2}\cos(2\theta_1 - 2\theta_2)$$

$$\tilde{V} = \frac{I_1 I_2}{2\omega_1\omega_2}\left[\cos 2\theta_1 + \cos 2\theta_2 + \frac{1}{2}\cos(2\theta_1 + 2\theta_2)\right]$$

Here $E_k = \dot{A}_k$; $I_k, \theta_k$ are the action-angle variables; $\bar{V}$, $V_r$ and $\tilde{V}$ are mean, low frequency (resonant), and high frequency (nonresonant) parts of perturbation, respectively, while the small perturbation parameter is the Hamiltonian itself: $\varepsilon = H \approx H_0 \sim I$; $V \sim \varepsilon H$ $(\omega_i \sim 1)$. This model describes the simplest ( $N = 2$ DF ) nontrivial case of the internal dynamics of a YM field.

In spite of degeneracy the KAM integrability still holds in this model under additional condition: $\omega_1/\omega_2 \neq p/q$ for integers $p$, $q$ satisfying $|p| + |q| \leq 4$ (see Ref. /2/). Yet, an interesting for the theory of YM fields case is just $\omega_1 = \omega_2$ when the exact resonance occurs at $H \to 0$. The crucial point is that there exists only one, single, resonance related to the low frequency perturbation term $V_r$ (4.1). In this case the averaged system ($\omega_1 = \omega_2 = 1$)

$$\langle H \rangle = H_0 + \bar{V} + V_r = I_1 + I_2 + \frac{I_1 I_2}{2}\left[1 + \frac{1}{2}\cos(2\theta_1 - 2\theta_2)\right] \tag{4.2}$$

is always globally integrable due to the specific resonant symmetry (only one of N linearly independent phase combinations is present in the Hamiltonian) which implies (N-1) resonant integrals (see, e.g., Ref. /6/). One of these integrals is always the unperturbed Hamiltonian, $H_0 = I_1 + I_2$ for system (4.2). Since $\langle H \rangle$ is also an integral the averaged perturbation $\langle V \rangle = \bar{V} + V_r = const$ as well. Two independent integrals determine phase curves of the averaged system (4.2) /27/:

$$J^2 = 1 - \frac{v}{1 + \frac{1}{2}\cos\varphi} \tag{4.3}$$

where $J = (I_1 - I_2)/H_0$; $\varphi = 2\theta_1 - 2\theta_2$, and $v = 8\langle V \rangle / H_0^2$. Thus, the motion structure is independent of $H_0$ in the limit $H_0 \to 0$, the $H_0$ value determining the motion time scale only. For example, the frequency of small oscillation around the stable periodic orbit $J = \varphi = 0$ ($v = 3/2$) is $\omega_0 = \sqrt{3/8}\, H_0$. There exists also a separatrix corresponding to the unstable periodic orbit $J = 0;\ \varphi = \pi$ ($v = 1/2$), and it is split by the high frequency perturbation $\tilde{V}$, the resonant small parameter being

$$\xi \sim exp\left(-\frac{C}{\varepsilon}\right) \tag{4.4}$$

where $\varepsilon \approx H_0 \sim \omega_0$, and $C \sim 1$, a numerical factor. Due to weak nonlinearity this latter parameter, and, hence, the stochastic layer width, is much smaller as compared to Eq. (3.1), while the resonance width is much bigger, and moreover is independent of $\varepsilon$ ($J_{max} = \sqrt{2/3}$) (4.3).

Thus, the model (4.1) is KAM integrable for $H \ll H_{cr} \sim C \sim 1$ (4.4). In Ref. /26/ $H_{cr} \approx 6.7$ was accepted from numerical simulation of this model. At $H \gg H_{cr}$ the motion is globally chaotic /26,27/, with small regular components incorporated though /27/.

At larger N the KAM integrability is generally destroyed as well, even in the limit $H_0 \to 0$. Consider /27/ the model of the type (4.1) with N = 3:



$$H_0 = \omega_1 I_1 + \omega_2 I_2 + \omega_3 I_3$$
$$2V = (A_1 A_2)^2 + (A_1 A_3)^2 + (A_2 A_3)^2 \tag{4.5}$$

If $\omega_1 = \omega_2 = \omega_3$ there are three resonances, and even the averaged system $\langle H \rangle$ is no longer integrable as the numerical simulation in Ref. /27/ confirms. Such a confirmation is always desirable to rule out any hidden symmetry in the system. The motion structure does again depend on the ratio $\langle V \rangle / H_0^2$ only, but not on $H_0$ as $H_0 \to 0$ (comp. Eq. (4.3)). According to numerical data in Ref. /27/ this structure includes both chaotic as well as regular (quasiperiodic) components of comparable measure.

## 5. The nature of dynamical chaos

A typical and the most important peculiarity of the chaotic motion is a fast (exponential, which is the fastest in a sense /11/) local instability that is divergence of a beam of close trajectories. This local dynamical behavior is described by the equations of motion linearized about one of chaotic trajectories. For a closed, time reversible system the instability is characterized by the $n$ Lyapunov exponents $\Lambda_i > 0$, the dimensionality of the chaotic component being (2n + 1) (see, e.g., /28/). Particularly, if the latter comprises the whole energy surface or a part of it, n = $N-1$.

Lyapunov exponents determine the metric entropy of a chaotic component /28/:

$$h = \sum_{i=1}^{n} \Lambda_i \geq \Lambda_m \tag{5.1}$$

which is also called sometimes the KS-entropy (Krylov-Kolmogorov-Sinai entropy). The maximal Lyapunov exponent $\Lambda_m$ is determined by

$$\Lambda_m = \lim_{t \to \infty} \frac{1}{t} \ln \frac{|\vec{\rho}(t)|}{|\vec{\rho}(0)|} \tag{5.2}$$

where $\vec{\rho}(t)$ is vector of the linearized solution. For almost any initial vector $\vec{\xi}(0)$ the solution $\vec{\xi}(t)$ is rapidly approaching the eigenvector related to $\Lambda_m$. This greatly simplifies the numerical procedure for calculating $\Lambda_m$. On the other hand, it is also sufficient to calculate $\Lambda_m$ only since all we need to know is whether $h > 0$. If so, almost all trajectories of the chaotic component are random according to the algorithmic theory of dynamical systems /5/.

In this theory the random means unpredictable, or uncomputable. It may be elucidated as follows. Consider a coarse-grained, or symbolic, trajectory which is a sequence of some integers $q_t$ $(0 \leq q_t < M)$ where $t$ is the integer time (of step T), and $M$ is the number of finite elements of the phase space partition. Each $q_t$ describes an instant position of the system to a finite accuracy. The latter is crucial, at least, for two reasons. First, it takes account of an unavoidable uncertainty of any observation or measurement in physics. Second, and the most important, it enables us to rigorously discern "simple" (regular) and "complicated" (chaotic) trajectories. Namely, the important concept of complexity, introduced by Kolmogorov for sequences, can be extended, in this way, on dynamical trajectories /5/. Loosely speaking, the complexity of a finite sequence is the amount of information necessary to reproduce this sequence. If the complexity is maximal, that is proportional to the sequence length, it is natural to define that sequence as random. For to reproduce such a sequence one actually needs to know it beforehand, either explicitly as given, or implicitly, as encoded somehow, in particular, in initial conditions of the motion. In the former case the reproduction is merely copying, and in the latter it is deciphering, in particular, by using the equations of motion. The crucial point of the algorithmic philosophy is impossibility to separate the two above cases in view of a continuous transition between them. For example, an intermediate step could be a change, say, from binary to decimal numbers or vice versa. Thus, a given random trajectory is just given and cannot be reproduced in any simpler way.

The principal result, due to Brudno, in the algorithmic theory of dynamical systems is /29,5/:



$$K = h \tag{5.3}$$

where $K$ stands for the mean Kolmogorov complexity of trajectories, per unit time. Hence, the origin of a chaotic trajectory maximal complexity and of its randomness lies in the initial conditions and, ultimately, in continuity of the phase space in classical mechanics.

The role of equations of motion themselves in producing random trajectories turns out to be secondary. It is to merely provide the local instability of motion and, hence, to grant dynamical significance to arbitrarily diminutive details of trajectory initial conditions. As such, the dynamical algorithm can be, hence, very simple that has appeared so puzzling until recently. Now we understand that simplicity of the system does actually eclipse the true origin of dynamical chaos.

Apparently simplest model of chaotic motion is described by the following one-dimensional mapping $\varphi \to \bar{\varphi}$ /34/

$$\bar{\varphi} = k\varphi \mod 1, \quad \text{or} \quad \bar{z} = z^k; \quad z = \exp(2\pi i \varphi) \tag{5.4}$$

where integer $k > 1$. The model represents dynamics of a single phase variable, and its motion is locally unstable ($h = \ln k$) and, hence, random. Nevertheless, the motion trajectory can be explicitly written down as

$$\varphi_t = \varphi_0 \cdot k^t \mod 1; \quad z_t = z_0^{(k^t)} \tag{5.5}$$

The randomness of sequence $\{\varphi_t\}$ ($t = 1, 2, \ldots$), for almost any initial phase $\varphi_0$, is due to taking the fractional part (mod 1) or else due to transition from the <u>turn angle</u> ($\bar{\varphi} - \varphi$) of the complex vector $z$ to the change in its <u>direction</u> (($\bar{\varphi} - \varphi$) mod 1). One may also say that a regular sequence $\{\varphi_0 \cdot k^t\}$ becomes random in respect to the period T of phase variable $\varphi$, the randomness depending on the arithmetic of number ($\varphi_0 / T$) (see below).

Let the dynamical space of system (5.4) (the interval [0,1]) be parted into M = k equal segments. If, moreover, the number $\varphi_0$ is given to the base $k$, that is as a finite or infinite sequence of integers $\{g_n\}$ $(0 \leq g_n < k)$, the map (5.4) acts as the shift of this sequence by one digit to the left. As a result, successive elements of symbolic trajectory $q_t = g_n$, while the whole trajectory can be represented by some single (irrational, generally) number $p$ $(0<p<1)$ which just coincides with the initial phase: $p = \varphi_0$.

This elementary example readily demonstrates that a random trajectory cannot be given in any a priori way since it is determined by the random $\varphi_0$ (having a random sequence of digits) which is uncomputable, and can be obtained a posteriori only, from observing the symbolic trajectory $p$. On the contrary, a regular, say, periodic symbolic trajectory can be given (computed) a priori but it does not determine the exact initial conditions.

A fundamental result in the algorithmic theory of randomness is in that almost all (in the Lebegue measure) real numbers turn out to be just random. Exceptional (nonrandom) ones are, of course, all the rational numbers as well as the computable irrationals like $\pi$, $e$ etc. In other words, a random number is a sort of "the thing in itself", it does exist, yet it cannot be produced in any way.

The following instructive example is described in Ref. /35/. Consider the uniformly rotating vector $z = \exp(2\pi i \nu t)$. Let us construct the mapping which fixes $z$ direction not in equal time steps, as usual, but at the instants $t_n$, such that

$$\nu t_n = \varphi_0 \cdot k^n \tag{5.6}$$

where integer $k > 1$. This mapping is equivalent to mapping (5.4), and, hence, for almost any $\varphi_0$, the sequence $\{z_n\}$ proves to be random even though the original continuous motion is regular, of course. The latter point led the authors of Ref. /35/ to the conclusion that such a randomness is just an "illusion". However, one hardly may accept that view. Instead, this example demonstrates again that a random sequence $\{t_n\}$, which converts the regular rotation into a random mapping, can be described by a deterministic algorithm (5.6). Note that for a given random $\varphi_0$ the sequence $\{t_n\}$ depends on the rotation frequency $\nu$. In other words, while the number $\varphi_0$ is random



absolutely, the sequence $\{t_n\}$ does so in respect to a given regular motion only. A true illusion, in this example, is apparent possibility of producing random $\{t_n\}$ which, in fact, cannot be done in any way as was explained above. Hence, that "induced" randomness will not occur for any a priori selection of the mapping instants $t_n$.

Another interesting question relates to the Poincaré mapping. Here the instants $t_n$ of trajectory crossings the surface of section are not given beforehand but, instead, are determined by the motion itself. What would be the relation between randomness of the continuous motion and that of the mapping? If the invariant measure is compact, i.e. the dynamic component comprises a finite domain of the phase space, the regular or chaotic oscillation gets steady. Particularly, it implies the existence of some time average for a trajectory recurrence to the surface of section: $T = \langle (t_{n+1} - t_n) \rangle$. Then, both KS-entropies, that of the mapping ($h_1$) and of continuous motion ($h$), are related as follows:

$$h_1 = Th \tag{5.7}$$

Hence, either both quantities are nonzero, and the motion is random or the both are zero, and the motion is nonrandom.

The concept of algorithmic randomness appears to be in conformity with our intuitive idea of what the random is like. Moreover, the developing of this concept has been actually guided by that conformity. At any event, that randomness does not mean the complete randomness, in particular, it does not exclude correlations in motion. It only implies some correlation decay, and, moreover, in both directions of time ($t \to \pm\infty$) in accordance with the dynamical reversibility.

For a given motion there is a continuous transition from deterministic to chaotic behavior. Generally, this transition can be characterized by the following randomness parameter:

$$r = \frac{Th}{\ln M} \tag{5.8}$$

the limit $r \to 0$ corresponding to an approximate and temporary imitation of deterministic evolution while the chaos is building up as $r \to \infty$.

The random, dynamically unstable motion posseses am important property of the statistical stability, i.e. stability of any averaged quantity, which, in turn, is a corollary of the structural stability of dynamical chaos (see, e.g., Ref. /11/).

On the other hand, the randomness of a motion does not fix its statistical properties. For example, the exponential local instability of motion, generally, does not imply any exponential relaxation, nor even the latter appears to be a typical case /30,20/. The reason is in different averaging for both quantities, the entropy and the correlation. In particular, the chaos border in the phase space, i.e. a coexistence of chaotic and regular components of motion, inevitably leads to a power-type relaxation ($\propto t^{-p}$; $p \sim 1$) /20/. Such a relaxation has been apparently observed both numerically /31,33/ and experimentally /32/ (see Ref. /20/). Note that a power correlation decay implies, generally, a fairly complicated and unusual statistical description of the dynamical chaos.

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References

1. V.I.Arnold and A.Avez, Ergodic Problems of Classical Mechanics, Benjamin (1968).

2. V.I.Arnold, Usp. Mat. nauk, 18, No 6 (1963) 91 (in Russian).

3. J.Moser, Stable and Random Motions in Dynamical Systems, Princeton Univ. Press (1973).

4. I.P.Kornfeld, Ya.G.Sinai, S.V.Fomin, Ergodic Theory, Nauka, 1980 (in Russian).

5. V.M.Alekseev and M.V.Yakobson, Physics Reports, 75 (1981) 287.

6. B.V.Chirikov, ibid, 52 (1979) 263.

7. G.M.Zaslavsky, ibid, 80 (1981) 157.

8. B.V.Chirikov, F.M.Izrailev, D.L.Shepelyansky, in Soviet Scientific Reviews, Section C, Vol. 2 (1981), p. 209.

9. A.J.Lichtenberg and M.A.Lieberman, Regular and Stochastic Motion, Springer-Verlag (1982).

10. A.V.Gaponov-Grekhov, M.I.Rabinovich, Priroda, 1981, No 2, p. 54 (in Russian).

11. Ya.G.Sinai, ibid, No. 3, p. 72.

12. B.V.Chirikov, ibid, 1982, No. 7, p. 15.

13. A.N.Kolmogorov, General Theory of Dynamical Systems and Classical Mechanics, Proc. Intern. Congress of Mathematicians, Amsterdam, 1954, p. 315.

14. H.Poincaré, Les méthodes nouvelles de la mécanique céleste, Paris, 1892-99.

15. J.Moser, Ann. Scuola Normale Sup., Pisa, ser. III, 20 (1966) 499.

16. F.Takens, Indag. Math. 33 (1971) 379.

17. V.I.Arnold, Dokl. Akad. Nauk SSSR 156 (1964) 9; The Problem of Stability and Ergodic Properties of Classical Dynamical Systems, Proc. Intern. Congress of Matematicians (Moscow, 1966), Moscow, Mir, 1968, p. 387 (in Russian).

18. N.N.Bogoliubov, Yu.A.Mitropolsky, A.M.Samoilenko, The Method of Rapid Convergence in Nonlinear Mechanics, Naukova dumka, Kiev, 1969 (in Russian).

19. E.Fermi, Phys. Z. 24 (1923) 261. Also, Collected Works of E.Fermi, Univ. of Chicago Press, 1965, Vol. I.

20. B.V.Chirikov, Chaotic Dynamics in Hamiltonian Systems with Divided Phase Space, 7th Sitges Conf. on Dynamical Systems and Chaos, Barcelona, 1982.

21. N.N.Bogoliubov and Yu.A.Mitropolsky, Asymptotic Methods in the Theory of Nonlinear Oscillations, Hindustan Publ. Corp., Dehli, 1961.

22. N.N.Nekhoroshev, Usp. mat. nauk, 32, No. 6 (1977) 5 (in Russian).

23. N.I.Gavrilov, Diff. uravneniya, 18 (1982) 219; 383 (in Russian).

24. S.L.Ziglin, Dokl. Akad. Nauk SSSR, 251 (1980) 786 (in Russian).

25. E.S.Nikolaevsky, L.N.Shchur, JETP (Pisma), 36 (1982) 176 (in Russian).

26. S.G.Matinyan, G.K.Savvidi, N.G.Ter-Arutyunyan, ibid, 34 (1981) 613.

27. B.V.Chirikov, D.L.Shepelyansky, Dynamics of Some Homogeneous Models for Classical Yang-Mills Fields, Preprint 82-120, Institute of Nuclear Physics, Novosibirsk, 1982 (in Russian).

28. Ya.B.Pesin, Usp. mat. nauk, 32, No. 4 (1977) 55 (in Russian).

29. A.A.Brudno, ibid, 33, No 1 (1978) 207.

30. Ya.G.Sinai, Instability and Stochasticity of Dynamical Systems, Preprint ITEP-91, Moscow, 1982 (in Russian).

31. B.V.Chirikov, D.L.Shepelyansky, Statistics of the Poincaré Recurrences, and the Structure of Stochastic Layer of a Nonlinear Resonance, Preprint INP-81-69, Novosibirsk, 1981 (in Russian).



32. D.Bora, P.I.John, Y.C.Saxena and R.K.Varma, Plasma Physics, 22 (1980) 653.

33. C.F.F.Karney, Long-Time Correlations in the Stochastic Regime, Physica D (to be published).

34. L.A.Rokhlin, Izv. Akad. Nauk SSSR, mat., 25 (1961) 499.

35. T.Erber, P.Johnson and P.Everett, Phys. Lett., 85A (1981) 61.