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B.G.Konopelchenko

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ON THE GENERAL STRUCTURE OF NONLINEAR
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B.G.Konopelchenko

Institute of Nuclear Physics,
630090, Novosibirsk 90, USSR

A b s t r a c t

The generalization of AKNS - technique to the two dimensional arbitrary order matrix spectral problem is given. The general form of the integrable equations and their Backlund transformations in 1+2 dimensions is found. The reduction problem is discussed.

I. Introduction

One of the main problem of the inverse scattering transform (IST) method is a problem of enumeration of the equations integrable by this method (see, e.g. [1,2]. The simple and convenient description of a class of partial differential equations integrable by the onedimensional second order bundle

$$\frac{\partial \psi}{\partial x} = \lambda A \psi + P(x,t)\psi \qquad (1.0)$$

has been given by AKNS[3]. Then this approach (AKNS-approach) has been generalized to the problem (1.0) of arbitrary order [4-9] and to some others one-dimensional spectral problems [9-11]. The infinitedimensional groups of Backlund transformations for these classes of integrable equations has been also found [12,7-9]. But up to now all the results obtained in the framework of AKNS-approach [3-12] are concerned to the equations in one spatial dimension.

The generalization of AKNS-method to the case of several spatial dimensions is of indubitable interest. The applicability of IST method to the multidimensional equations has been demonstrated in Refs. [13,14]. Various concrete twodimensional and multidimensional evolution equations have been considered [13-18]. Multidimensional spectral problems possesses a number of specific features. Nevertheless, as we shall see, the technique described in Refs. [7-9] permit a generalisation to the 1+2 dimensions (one time and two spatial dimensions) case.

In the present paper we consider twodimensional spectral problem of the form

$$\frac{\partial \psi}{\partial x} + A\frac{\partial \psi}{\partial y} = P(x,y,t)\psi \qquad (1.1)$$

where A is an arbitrary constant semisimple matrix (i.e. a diagonal matrix), potential $P(x,y,t)$ is a matrix N x N such that $P(x,y,t) \xrightarrow[R=\sqrt{x^2+y^2}\to\infty]{} 0$. The order N of matrix problem (1.1) is arbitrary one. Spectral problem (1.1) is a natural twodimensional generalisation of the onedimensional bundle (1.0). Spectral problems of the type (1.1) (with diagonal matrix A) and some concrete equations connected with it, have been considered earlier in

Refs. [13-15,17].

In the present paper we find the general form of nonlinear evolution equations in 1+2 dimensions integrable by the problem (1.1). We also construct the universal infinitedimensional group of Backlund transformations and infinitedimensional symmetry group for these equations. The reduction problem for general equations and some concrete reductions are considered too. The results obtained are generalization to the two spatial dimensions of the corresponding results for bundle (1.0) (see [7,8]). We want to note that this 1+2 - dimensional generalization is a nontrivial one and possesses various interesting features.

The paper is organized as follows. In the second section we introduce some special solutions of linear problem (1.1), scattering matrix and obtain several important relations. In the third section we calculate the recursion operators $\hat{\Lambda}_{B;A}$, $\check{\Lambda}_{B;A}$ which play a fundamental role in our constructions. The general form of Backlund transformations and integrable equations is found in section 4. In the fifth section the integrals of motion are calculated. The reduction problem is discussed in the section 6.

II. Some preliminary relations

We will assume that potential $P(x,y,t) \to 0$ at $R = \sqrt{x^2+y^2} \to \infty$ so that it guarantee the existence of all the integrals which will be appear in our calculations and that $\int_{-\infty}^{+\infty} dy \frac{\partial}{\partial y} (\dots) = 0$.

We will also assume that potential satisfy the gauge condition $P_0 = 0$ where P_0 is a projection of potential P onto

g_0 - component of the Fitting decomposition with respect to A. Let us recall shortly it properties (see e.q. [19]). Fitting decomposition of the general linear matrix algebra $gl(N,C)$ with respect to semisimple matrix A is a decomposition into the tensor sum $gl(N,C) = g_0 \oplus g_F$ where g_0 is a subalgebra of matrixes commuting with A ($g_0 = \{g \in gl(N,C), [g,A]=0\}$) and g_F is a tensor sum of nonzero root subspaces. Then $[g_0, g_0] \subset g_0$ and $[g_0, g_F] \subset g_F$. For arbitrary matrix B of the order N we have a simple decomposition $B = B_0 + B_F$ where B_0 is a projection of B onto g_0 and B_F is a projection of B onto g_F. For potential we have the decomposition $P(x,y,t) = P_0(x,y,t) + P_F(x,y,t)$ too. Using the invariance of the problem (1.1) under the transformations

$$\Psi \to \Psi' = G(x,y)\Psi, \quad P \to P' = GPG^{-1} + \left(\frac{\partial G}{\partial x} + A\frac{\partial G}{\partial y}\right)G^{-1}$$

where $G = G_0$, it is always possible to achieve that $P_0 = 0$. A sense of the gauge $P_0 = 0$ consists in the excluding of pure gauge (nondynamical) degrees of freedom from $P(x,y,t)$.

Let us consider now the linear problem (1.1). We will denote the solutions of this problem by Ψ. Let us introduce follows to Ref. [15] matrixes-solutions $\hat{F}^+_\lambda(x,y)$ and $\hat{F}^-_\lambda(x,y)$ of problem (1.1) given by their asymptotic behaviour

$$\hat{F}^+_\lambda(x,y) \xrightarrow[x \to +\infty]{} (2\pi i)^{-\frac{1}{2}} e^{\lambda(y-Ax)}, \quad \hat{F}^-_\lambda(x,y) \xrightarrow[x \to -\infty]{} (2\pi i)^{-\frac{1}{2}} e^{\lambda(y-Ax)}$$

and scattering matrix $\hat{S}(\tilde{\lambda}, \lambda, t)$:

$$\hat{F}^+_\lambda(x,y,t) = \int_{-\infty}^{+\infty} d\tilde{\lambda}\, \hat{F}^-_{\tilde{\lambda}}(x,y,t)\, S(\tilde{\lambda}, \lambda, t).$$

Let us consider also the adjoint to (1.1) problem

$$\frac{\partial \check{\Psi}}{\partial x} + \frac{\partial \check{\Psi}}{\partial y} A = -\check{\Psi} P(x,y,t). \qquad (2.1)$$

We introduce the matrix-solutions $\check{F}^+_\lambda(x,y)$ and $\check{F}^-_\lambda(x,y)$ of the problem (2.1)

$$\check F_\lambda^+(x,y)\xrightarrow[x\to+\infty]{}(2\pi i)^{-\frac12}e^{-\lambda(y-Ax)},\quad \check F_\lambda^-(x,y)\xrightarrow[x\to-\infty]{}(2\pi i)^{-\frac12}e^{-\lambda(y-Ax)}$$

and corresponding scattering matrix $\check S(\lambda,\tilde\lambda,t)$:

$$\check F_\lambda^+(x,y,t)=\int_{-\infty}^{+\infty}d\tilde\lambda\,\check S(\lambda,\tilde\lambda,t)\,F_{\tilde\lambda}^-(x,y,t).$$

It isn't difficult to show that the following relations hold

$$\int_{-\infty}^{+\infty}dy\,\check F_{\tilde\lambda}^{\pm}(x,y,t)\hat F_\lambda^{\pm}(x,y,t)=\delta(\tilde\lambda-\lambda), \tag{2.2}$$

$$\int_{-\infty}^{+\infty}d\lambda\,\hat F_\lambda^{\pm}(x,y,t)\,\check F_\lambda^{\pm}(x,y',t)=\delta(y'-y), \tag{2.3}$$

$$\int_{-\infty}^{+\infty}d\mu\,\check S(\tilde\lambda,\mu,t)\,\hat S(\mu,\lambda,t)=\delta(\tilde\lambda-\lambda) \tag{2.4}$$

where $\delta(\lambda)$ is a Dirac delta-function.

Let now P and P' are two different potentials and $\hat\Psi$, $\hat\Psi'$, $\check\Psi$, $\check\Psi'$ are corresponding solutions of the problems (1.1) and (2.1). Using (1.1), (2.1) and taking into account (2.3) one can show that

$$\hat\Psi'_\lambda(x,y)-\int_{-\infty}^{+\infty}d\tilde\lambda\,\hat\Psi_{\tilde\lambda}(x,y)K(\tilde\lambda,\lambda)= \tag{2.5}$$

$$=-\int_{-\infty}^{+\infty}d\tilde\lambda\,\hat\Psi_{\tilde\lambda}(x,y)\int_x^{\infty}dz\int_{-\infty}^{+\infty}dy'\,\check\Psi_{\tilde\lambda}(z,y')(P'(z,y')-P(z,y'))\hat\Psi'_\lambda(z,y')$$

where $K(\tilde\lambda,\lambda)=\int_{-\infty}^{+\infty}dy\,\check\Psi_{\tilde\lambda}(x,y)\hat\Psi'_\lambda(x,y)\big|_{x=+\infty}$.

Putting $\Psi'_\lambda=\hat F^+_\lambda$ in (2.5) and proceeding to the limit $x\to-\infty$ one gets

$$\hat S'(\tilde\lambda,\lambda)-\hat S(\tilde\lambda,\lambda)= \tag{2.6}$$

$$=-\int_{-\infty}^{+\infty}d\mu\,\hat S(\tilde\lambda,\mu)\int_{-\infty}^{+\infty}dx\,dy\,\check F_\mu^+(x,y)(P'(x,y)-P(x,y))\hat F_\lambda^{+\prime}(x,y).$$

Formula (2.6) which relates a change of the potential P to that of the scattering matrix plays a fundamental role in further considerations.

The mapping $P(x,y,t)\to S(\tilde\lambda,\lambda,t)$ given by spectral problem (1.1) determine a correspondence between the transformations $P\to P'$ on the manifold of potentials $\{P(x,y,t)\}$ and the transformations $S\to S'$ on the manifold of scattering matrixes $\{S(\tilde\lambda,\lambda,t)\}$. This fact follows from diagram

$$\begin{array}{ccc} P & \xrightarrow{(1.1)} & S \\ T_P\downarrow & & \downarrow T_S \\ P' & \xrightarrow{(1.1)} & S' \end{array}.$$

Let us now consider only such transformations T that

$$\hat S(\tilde\lambda,\lambda,t)\xrightarrow{T_s}\hat S'(\tilde\lambda,\lambda,t)=B^{-1}(\tilde\lambda,t)\hat S(\tilde\lambda,\lambda,t)C(\lambda,t) \tag{2.7}$$

where $B(\tilde\lambda,t)$ and $C(\lambda,t)$ are some (in general, arbitrary) matrixes commuting with A, i.e. $B=B_0$, $C=C_0$. The "restricted" transformations of the type (2.7) are, as we shall see, wide enough.

Combining the relation (2.6) with (2.7) and taking into account (2.4) one find

$$\int_{-\infty}^{+\infty}d\mu\,\check S(\tilde\lambda,\mu)(1-B(\mu,t))\hat S'(\mu,\lambda)+C(\lambda,t)\delta(\tilde\lambda-\lambda)=$$

$$=-\int_{-\infty}^{+\infty}dx\,dy\,\check F^+_{\tilde\lambda}(x,y)(P'(x,y)-P(x,y))\hat F^{+\prime}_\lambda(x,y). \tag{2.8}$$

Then one can prove the following identity

$$\int_{-\infty}^{+\infty}d\mu\,\check S(\tilde\lambda,\mu)(1-B(\mu,t))\hat S'(\mu,\lambda)-\delta(\tilde\lambda-\lambda)(1-B(\lambda,t))= \tag{2.9}$$

$$= -\int_{-\infty}^{+\infty} dy\, \check{F}_{\tilde{\lambda}}^{+}(x,y)\left(1 - B\left(\frac{\partial}{\partial y}, t\right)\right)\hat{F}_{\lambda}^{+\prime}(x,y)\Big|_{x=-\infty}^{x=+\infty} = \tag{2.9}$$

$$= \int_{-\infty}^{+\infty} dx\,dy\, \check{F}_{\tilde{\lambda}}^{+}(x,y)\Big(P(x,y)\left(1 - B\left(\frac{\partial}{\partial y}, t\right)\right)\hat{F}_{\lambda}^{+\prime}(x,y) -$$

$$- \left(1 - B\left(\frac{\partial}{\partial y}, t\right)\right) P'(x,y)\, \hat{F}_{\lambda}^{+\prime}(x,y)\Big).$$

Taking the projection of the equality (2.8) onto g_F and using the identity (2.9) we obtain

$$\int_{-\infty}^{+\infty} dx\,dy \Big\{ \check{F}_{\tilde{\lambda}}^{+}(x,y)\Big(B\left(\frac{\partial}{\partial y}, t\right) P'(x,y)\, \hat{F}_{\lambda}^{+\prime}(x,y) -$$

$$- P(x,y)\, B\left(\frac{\partial}{\partial y}, t\right) \hat{F}_{\lambda}^{+\prime}(x,y)\Big)\Big\}_F = 0. \tag{2.10}$$

The matrix $B\left(\frac{\partial}{\partial y}, t\right)$ which is contained in formulae (2.10) can be represented in the form $B\left(\frac{\partial}{\partial y}, t\right) = \sum_{\alpha=1}^{r_A} B_\alpha\left(\frac{\partial}{\partial y}, t\right) H_\alpha$ where matrixes H_α $(\alpha = 1, \ldots, r_A)$ form a basis of subalgebra g_0 and $B_\alpha\left(\frac{\partial}{\partial y}, t\right)$ are some functions. Below we will consider only the entire functions $B_\alpha\left(\frac{\partial}{\partial y}, t\right)$, i.e. $B_\alpha\left(\frac{\partial}{\partial y}, t\right) = \sum_{n=0}^{\infty} b_{\alpha n}(t)\left(\frac{\partial}{\partial y}\right)^n$ where $b_{\alpha n}(t)$ are arbitrary functions. For such functions $B_\alpha\left(\frac{\partial}{\partial y}, t\right)$ the equality (2.10) can be rewritten as follows

$$\int_{-\infty}^{+\infty} dx\,dy \sum_{\alpha=1}^{r_A}\sum_{n=0}^{\infty} b_{\alpha n}(t)\, tr\Big(H_\alpha P'(x,y)(-1)^n\, \check{\Phi}_{(n)F}^{F} -$$

$$- P(x,y)\, H_\alpha\, \hat{\Phi}_{(n)F}^{F}\Big) = 0 \tag{2.11}$$

where tr is a usual matrix trace and

$$\left(\hat{\Phi}_{(n)}^{(im)}\right)_{ke} \overset{def}{=} \frac{\partial^n (\hat{F}_{\lambda}^{+})'_{km}}{\partial y^n}\left(\check{F}_{\tilde{\lambda}}^{+}\right)_{il}, \tag{2.12}$$

$$\left(\check{\Phi}_{(n)}^{(im)}\right)_{ke} \overset{def}{=} \left(\hat{F}_{\lambda}^{+}\right)'_{km} \frac{\partial^n (\check{F}_{\tilde{\lambda}}^{+})_{ie}}{\partial y^n} \tag{2.12}$$

$(n = 0, 1, 2, \ldots).$

III. The recursion operators

For further transformations of the equality (2.11) one must establish the relations between the quantities $\Phi_{(n)F}^{F}$ with different n.

Let us firstly consider quantity $\hat{\Phi}_{(n)F}$. Let us differentiate n - 1 times over variable y the system (1.1) for $(\hat{F}_{\lambda}^{+})'_{km}$ and then multiple the obtained equation by $\check{F}_{ie}^{+}$. Forming in the obtained equality the total derivatives over x and y and taking into account (2.1) one can obtain the equation

$$\frac{\partial \hat{\Phi}_{(n-1)}}{\partial x} + \frac{\partial \hat{\Phi}_{(n-1)}}{\partial y} A = -\left[A, \hat{\Phi}_{(n)}\right] +$$

$$+ P'\hat{\Phi}_{(n-1)} - \hat{\Phi}_{(n-1)} P + \sum_{m=0}^{n-2} C_m^{n-1} P'_{(n-1-m)} \hat{\Phi}_{(m)} \tag{3.1}$$

where $C_m^n = \frac{n!}{m!(n-m)!}$ and $P_{(k)} \overset{def}{=} \frac{\partial^k P(x,y)}{\partial y^k}$. Then if one project the equation (3.1) onto g_0 and use the relation $(\Phi\, \Psi_F)_0 = (\Phi_F\, \Psi_F)_0$ one get

$$\frac{\partial \hat{\Phi}_{(n-1)0}}{\partial x} + \frac{\partial \hat{\Phi}_{(n-1)0}}{\partial y} A =$$

$$= \Big(P'\hat{\Phi}_{(n-1)F} - \hat{\Phi}_{(n-1)F} P + \sum_{m=0}^{n-2} C_m^{n-1} P'_{(n-1-m)} \hat{\Phi}_{(m)F}\Big)_0. \tag{3.2}$$

The integration of (3.2) give

$$\hat{\Phi}_{(n-1)0}(x,y) = \hat{\Phi}_{(n-1)0}(x = +\infty, y) - \tag{3.3}$$

$$-\mathcal{Y}^{-}\Big(P'\hat{\Phi}_{(n-1)F}-\hat{\Phi}_{(n-1)F}P+\sum_{m=0}^{n-2}C_m^{n-1}P'_{(n-1-m)}\hat{\Phi}_{(m)F}\Big)_0 \quad (3.3)$$

where for $f_0(x,y)=\sum_{\alpha=1}^{r_A} f_\alpha(x,y)H_\alpha$ and $A=\sum_{\alpha=1}^{r_A} a_\alpha H_\alpha$

$$(\mathcal{Y}^{-}f_0)(x,y)\overset{def}{=\!=}\sum_{\alpha=1}^{r_A}H_\alpha\int_x^\infty dz\, f_\alpha(z,\, a_\alpha(z-x)+y).$$

So one can express the quantities $\hat{\Phi}_{(n)0}$ though the quantities $\hat{\Phi}_{(m)F}$ $(m=0,1,\dots,n-1)$. As a result taking the projection of the equation (3.1) onto subspance $\mathcal{G}_F$, taking into account the equality (3.3) and relations $(\Phi_0\Psi_F)_F=\Phi_0\Psi_F$, $\hat{\Phi}^{(F)}_{(n)0}(x\to+\infty,y)=0$ we obtain

$$\big[A,\hat{\Phi}^F_{(n)F}\big]=\hat{\Lambda}_{(1)}\hat{\Phi}^{(F)}_{(n-1)F}+$$

$$+\sum_{m=0}^{n-2}C_m^{n-1}\Big\{\big(P'_{(n-1-m)}\hat{\Phi}^F_{(m)F}\big)_F-P'\mathcal{Y}^{-}\big(P'_{(n-1-m)}\hat{\Phi}^{(F)}_{(m)F}\big)_0+$$

$$(3.4)$$

$$+\mathcal{Y}^{-}\big(P'_{(n-1-m)}\hat{\Phi}^{(F)}_{(m)F}\big)_0\cdot P-P'_{(n-1-m)}\mathcal{Y}^{-}\big(P'\hat{\Phi}^{(F)}_{(m)F}-\hat{\Phi}^{(F)}_{(m)F}P\big)_0-$$

$$-\sum_{e=0}^{m-1}C_e^m P'_{(n-1-m)}\mathcal{Y}^{-}\big(P'_{(m-e)}\hat{\Phi}^{(F)}_{e(F)}\big)_0\Big\}$$

$$(n=1,2,3,\dots)$$

where operator $\hat{\Lambda}_{(1)}$ acts as follows

$$\hat{\Lambda}_{(1)}\Phi=-\frac{\partial\Phi}{\partial x}-\frac{\partial\Phi}{\partial y}A+(P'\Phi-\Phi P)_F- \quad (3.5)$$

$$-P'\mathcal{Y}^{-}(P'\Phi-\Phi P)_0+\mathcal{Y}^{-}(P'\Phi-\Phi P)_0\cdot P$$

From the relations (3.4) it follows that there exist such operators $\hat{\Lambda}_{(n)A}$ that

$$\hat{\Phi}^{(F)}_{(n)F}=\hat{\Lambda}_{(n)A}\Phi^{(F)}_{(0)F}\qquad(n=1,2,3,\dots)\qquad(3.6)$$

where $\hat{\Lambda}_{(n)A}\Phi\overset{def}{=\!=}(\hat{\Lambda}_{(n)}\Phi)_A$, $[A,\Phi_A]\overset{def}{=\!=}\Phi$ and $(\Phi^{(m)}_{(0)})_{ke}\overset{def}{=\!=}(\hat{F}^+)'_{km}(\check{F}^+)_{ie}$. The operators $\hat{\Lambda}_{(n)A}$ are determined by the following recursion relations:

$$\hat{\Lambda}_{(n)A}\Phi=\hat{\Lambda}_{(1)A}\hat{\Lambda}_{(n-1)A}\Phi+$$

$$+\sum_{m=0}^{n-2}C_m^{n-1}\Big\{\big(P'_{(n-1-m)}\hat{\Lambda}_{(m)A}\Phi\big)_F-P'\mathcal{Y}^{-}\big(P'_{(n-1-m)}\hat{\Lambda}_{(m)A}\Phi\big)_0+$$

$$(3.7)$$

$$+\mathcal{Y}^{-}\big(P'_{(n-1-m)}\hat{\Lambda}_{(m)A}\Phi\big)_0\cdot P-P'_{(n-1-m)}\mathcal{Y}^{-}\big(P'\hat{\Lambda}_{(m)A}\Phi-\hat{\Lambda}_{(m)A}\Phi\cdot P\big)_0-$$

$$-\sum_{e=0}^{m-1}C_e^m P'_{(n-1-m)}\mathcal{Y}^{-}\big(P'_{(m-e)}\hat{\Lambda}_{(e)A}\Phi\big)_0\Big\}_A\quad(n=2,3,\dots).$$

In a similar way one can show that

$$\check{\Phi}^{(F)}_{(n)F}=\check{\Lambda}_{(n)A}\Phi^{(F)}_{(0)F}\qquad(n=1,2,3,\dots)\qquad(3.8)$$

Operators $\check{\Lambda}_{(n)A}$ are determined by the recursion relations

$$\check{\Lambda}_{(n)A}\Phi=\check{\Lambda}_{(1)A}\check{\Lambda}_{(n-1)A}\Phi+$$

$$+\sum_{m=0}^{n-2}C_m^{n-1}\Big\{\big(\check{\Lambda}_{(m)A}\Phi\cdot P_{(n-1-m)}\big)_F-P'\mathcal{Y}^{-}\big(\check{\Lambda}_{(m)A}\Phi\cdot P_{(n-1-m)}\big)_0+$$

$$(3.9)$$

$$+\mathcal{J}^-(\check{\Lambda}_{(m)A}\Phi\cdot P_{(n-1-m)})_0\cdot P - \mathcal{J}^-(P'\check{\Lambda}_{(m)A}\Phi - \check{\Lambda}_{(m)A}\Phi\cdot P)_0\cdot P_{(n-1-m)}$$

$$+\sum_{e=0}^{m-1} C_e^m \mathcal{J}^-(\check{\Lambda}_{(e)A}\Phi\cdot P_{(m-e)})_0\cdot P_{(n-1-m)}\Big\}_A \qquad (3.9)$$

$$(n=2,3,\dots)$$

where

$$\check{\Lambda}_{(1)}\Phi = \frac{\partial\Phi}{\partial x} + A\frac{\partial\Phi}{\partial y} - (P'\Phi - \Phi P)_F + P'\mathcal{J}^-(P'\Phi - \Phi P)_0 - \mathcal{J}^-(P'\Phi - \Phi P)_0\cdot P \qquad (3.10)$$

The operators $\hat{\Lambda}_{(n)A}$ and $\check{\Lambda}_{(n)A}$ are not independent. Throwing the derivative $\frac{\partial}{\partial y}$ from $\check{F}^+$ to $\hat{F}^{+\prime}$, for example, in the quantity $\Phi_{(n)}$ it is not difficult to show that

$$\check{\Lambda}_{(n)A} = \sum_{k=0}^{n}(-1)^k C_k^n \frac{\partial^{n-k}}{\partial y^{n-k}}\hat{\Lambda}_{(k)A}. \qquad (3.11)$$

In the further constructions we will use also the operators $\check{\Lambda}^+_{(n)A}$, $\hat{\Lambda}^+_{(n)A}$ adjoint to the operators $\check{\Lambda}_{(n)A}$, $\hat{\Lambda}_{(n)A}$ with respect to the bilinear form $\langle \chi, \Phi\rangle = \int_{-\infty}^{+\infty} dxdy\, tr(\chi_F(x,y)\,\Phi_F(x,y))$. The corresponding recursion relation, for example, for operators $\hat{\Lambda}^+_{(n)A}$ are of the form

$$\hat{\Lambda}^+_{(n)A}\Phi = \hat{\Lambda}^+_{(n-1)A}\hat{\Lambda}^+_{(1)A}\Phi - \sum_{m=0}^{n-2}C_m^{n-1}\hat{\Lambda}^+_{(m)A}\Big\{(\Phi_A P'_{(n-1-m)})_F - \mathcal{J}^+(\Phi_A P' - P\Phi_A)_0\cdot P'_{(n-1-m)} - \mathcal{J}^+(\Phi_A P'_{(n-1-m)})_0\cdot P' + P\cdot\mathcal{J}^+(\Phi_A P'_{(n-1-m)})_0\Big\} + \sum_{m=0}^{n-2}C_m^{n-1}\sum_{e=0}^{m-1}C_e^m\hat{\Lambda}^+_{(e)A}\mathcal{J}^+(\Phi_A P'_{(n-1-m)})_0\cdot P'_{(m-e)} \qquad (3.12)$$

where $\hat{\Lambda}^{+}_{(n)A}\Phi = -\hat{\Lambda}^+_{(n)}\Phi_A$ and

$$\hat{\Lambda}^+_{(1)}\Phi = \frac{\partial\Phi}{\partial x} + A\frac{\partial\Phi}{\partial y} + (\Phi P' - P\Phi)_F + P\mathcal{J}^+(\Phi P' - P\Phi)_0 - \mathcal{J}^+(\Phi P' - P\Phi)_0\cdot P' \qquad (3.13)$$

where

$$(\mathcal{J}^+ f_0)(x,y) \stackrel{def}{=} \sum_{\alpha=1}^{r_A} H_\alpha \int_{-\infty}^{x} dz\, f_\alpha(z, a_\alpha(z-x)+y).$$

The operators $\check{\Lambda}^+_{(n)A}$ can be determined from the recursion relations analogous to (3.12) or from the relations

$$\check{\Lambda}^+_{(n)A} = (-1)^n\sum_{k=0}^{n} C_k^n\,\hat{\Lambda}^+_{(k)A}\frac{\partial^{n-k}}{\partial y^{n-k}}. \qquad (3.14)$$

Analogously to (3.11) and (3.14) one can express the operators $\hat{\Lambda}_{(n)A}$ and $\hat{\Lambda}^+_{(n)A}$ through respectively $\check{\Lambda}_{(m)A}$ and $\check{\Lambda}^+_{(m)A}$.

IV. General structure of the integrable equations and Backlund-transformations

The existence of the recursion operators of the type $\Lambda_{(n)A}$ and $\hat{\Lambda}^+_{(n)A}$ is extremely important in the generalized AKNS-method. In our case the relations (3.6) and (3.8) allow to rewrite the equality (2.11) in the form

$$\int_{-\infty}^{+\infty} dxdy \sum_{\alpha=1}^{r_A}\sum_{n=0}^{\infty} b_{\alpha n}(t)\, tr\big(H_\alpha P'(x,y)(-1)^n\check{\Lambda}_{(n)A}\Phi^{(F)}_{(0)F} - P(x,y)H_\alpha\hat{\Lambda}_{(n)A}\Phi^{(F)}_{(0)F}\big) = 0 \qquad (4.1)$$

where $\hat{\Lambda}_{(0)} = \check{\Lambda}_{(0)} \equiv 1$.

From (4.1) we have

$$\int_{-\infty}^{+\infty} dx\,dy\; tr\Big(\Phi^{(F)}_{(0)F}(x,y)\Big\{\sum_{\alpha=1}^{r_A}\sum_{n=0}^{\infty} b_{\alpha n}(t)\big((-1)^n\check{\Lambda}^{+}_{(n)A} H_\alpha P' - \hat{\Lambda}^{+}_{(n)A} P H_\alpha\big)\Big\}\Big) = 0 \qquad (4.2)$$

where operators $\hat{\Lambda}^{+}_{(n)A}$ and $\check{\Lambda}^{+}_{(n)A}$ are given by formulas (3.12)-(3.14).

The equality (4.2) is fulfilled if

$$\sum_{\alpha=1}^{r_A}\big(B_\alpha(\check{\Lambda}^{+}_{A},t)\,H_\alpha P' - B_\alpha(\hat{\Lambda}^{+}_{A},t)\,PH_\alpha\big)=0 \qquad (4.3)$$

where

$$B_\alpha(\check{\Lambda}^{+}_{A},t) \overset{def}{=} \sum_{n=0}^{\infty}(-1)^n b_{\alpha n}(t)\,\check{\Lambda}^{+}_{(n)A},$$

$$B_\alpha(\hat{\Lambda}^{+}_{A},t) \overset{def}{=} \sum_{n=0}^{\infty} b_{\alpha n}(t)\,\hat{\Lambda}^{+}_{(n)A}.$$

If quantities $\Phi^{(F)}_{(0)F}(x,y)$ form the complete set (similar to the onedimensional case) then the equality (4.3) is also a necessary condition of fulfilment of the equality (4.2).

The relation (4.3) just determined the transformation of the potential $P \to P'$ which corresponds to the transformation of the scattering matrix $S \to S'$ of the form (2.7). It is important that the relation (4.3) contains only the potential P and transformed potential P'. We restricted overselfs by the transformation law of scattering matrix of the form (2.7) just in order that it will be possible to convert the transformation law of scattering matrix into the explicit transformation law of potential which contains only P and P'.

The transformations (2.7), (4.3) form, as it is easy to see from (2.7), an infinitedimensional group. If A is regular matrix (i.e. all eigenvalues of A are different) then subalgebra $\mathcal{G}_0$ is abelian. In this case the group of transformations (2.7), (4.3) is infinitedimensional abelian group.

The structure of this group of transformations (2.7), (4.3) (group B) is determined by the spectral problem (1.1). Group B which acts on the manifold of the potentials $\{P(x,y,t)\}$ and on the manifold of the scattering matrixes $\{S(\tilde{\lambda},\lambda,t)\}$ play a fundamental role in the analysis of nonlinear systems connected with the problem (1.1) and their properties.

Let us consider a one-parameter subgroup of this group given by

$$B = \sum_{\alpha=1}^{r_A} e^{-\int_t^{t'} ds\,\Omega_\alpha(\lambda,s)} H_\alpha \qquad (4.4)$$

where $\Omega_\alpha(\lambda,t)$ are some functions entire on λ $\big(\Omega_\alpha(\lambda,t) = \sum_{n=0}^{\infty}\omega_{\alpha n}(t)\lambda^n\big)$ and C = B.

It is not difficult to see that transformation (2.7) with matrix B of the form (4.4) is a displacement in time t:

$$\hat{S}(\tilde{\lambda},\lambda,t) \longrightarrow \hat{S}'(\tilde{\lambda},\lambda,t) = B^{-1}S(\tilde{\lambda},\lambda,t)B = S(\tilde{\lambda},\lambda,t') \qquad (4.5)$$

The corresponding transformation of potential is $P(x,y,t) \to P'(x,y,t) = P(x,y,t')$ and is given by formula*

$$\sum_{\alpha=1}^{r_A}\Big(e^{-\int_t^{t'} ds\,\Omega_\alpha(\check{\Lambda}^{+}_{A},s)} H_\alpha P(x,t') - e^{-\int_t^{t'} ds\,\Omega_\alpha(\hat{\Lambda}^{+}_{A},s)} P(x,t) H_\alpha\Big) = 0 \qquad (4.6)$$

where in the operators $\check{\Lambda}^{+}_{(n)A}$ and $\hat{\Lambda}^{+}_{(n)A}$ one put $P'(x,y,t) = P(x,y,s)$ and $\Omega_\alpha(\check{\Lambda}^{+}_{A},t) \overset{def}{=} \sum_{n=0}^{\infty}\omega_{\alpha n}(t)(-1)^n\check{\Lambda}^{+}_{(n)A}$ and $\Omega_\alpha(\hat{\Lambda}^{+}_{A},t) \overset{def}{=} \sum_{n=0}^{\infty}\omega_{\alpha n}(t)\,\hat{\Lambda}^{+}_{(n)A}$.

* Transformations of the form (4.6) was considered for the first time in Ref. [12] for onedimensional bundle (1.0) at N = 2, $A=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$.

At fixed functions $\Omega_\alpha(\lambda,t)$ the one-parameter group of transformations (4.6) determine in inexplicit form the flow $Y_\Omega : P(x,y,t) \to P(x,y,t')$, in other words, an evolution system. This evolution system can be also described by some nonlinear evolution equation.

Indeed let us consider the infinitesimal displacement in time: $t \to t' = t+\varepsilon$, $\varepsilon \to 0$. In this case $P(x,y,t') = P(x,y,t) + \varepsilon \frac{\partial P(x,y,t)}{\partial t}$ and $B_\alpha(\lambda,t) = 1 - \varepsilon\,\Omega_\alpha(\lambda,t)$. Substituting these expressions into (4.6) and keeping the terms of the first order on ε we obtain an evolution equation

$$\frac{\partial P(x,y,t)}{\partial t} - \sum_{\alpha=1}^{r_A}\left(\Omega_\alpha(\check{L}^+_A,t)H_\alpha P - \Omega_\alpha(\hat{L}^+_A,t)PH_\alpha\right) = 0 \qquad (4.7)$$

where $\hat{L}^+_{(m)A} \overset{def}{=} \hat{\Lambda}^+_{(m)A}(P'=P)$ and $\check{L}^+_{(m)A} \overset{def}{=} \check{\Lambda}^+_{(m)A}(P'=P)$. The operators $\hat{L}^+_{(m)A}$ are calculated from the recursion relations (3.12) at P' = P. For example,

$$\hat{L}^+_{(1)}\cdot = \frac{\partial}{\partial x} + A\frac{\partial}{\partial y} - [P,\cdot]_F - [P, \mathcal{Y}^+[P,\cdot]_0], \qquad (4.8)$$

$$\hat{L}^+_{(2)A}\varphi = (\hat{L}^+_{(1)A})^2\varphi - (\varphi_A P_{(1)})_F - \mathcal{Y}^+([P,\varphi_A])_0 \cdot P_{(1)} - [P, \mathcal{Y}^+(\varphi_A P_{(1)})_0]. \qquad (4.9)$$

The operators $\check{L}^+_{(m)A}$ can be calculated by the formula

$$\check{L}^+_{(n)A} = (-1)^n \sum_{k=0}^{n} C^n_k \, \hat{L}^+_{(k)A} \frac{\partial^{n-k}}{\partial y^{n-k}}$$

For the scattering matrix from (2.7) we correspondingly obtain the following evolution equation

$$\frac{d\hat{S}(\tilde{\lambda},\lambda,t)}{dt} = Y(\tilde{\lambda},t)\,\hat{S}(\tilde{\lambda},\lambda,t) - \hat{S}(\tilde{\lambda},\lambda,t)\,Y(\lambda,t) \qquad (4.10)$$

where

$$Y(\lambda,t) \overset{def}{=} \sum_{\alpha=1}^{r_A} \Omega_\alpha(\lambda,t) H_\alpha . \qquad (4.11)$$

The nonlinear evolution equation (4.7) determine the flow $Y_\Omega : P(x,y,t) \to P(x,y,t')$ in the infinitesimal form. The relation (4.6) which does not contain the derivative $\frac{\partial P}{\partial t}$ is an "integrated" form of the evolution equation (4.7). Class of the equations (4.7) is characterized by integer N, operators $\hat{L}^+_{(m)A}$, $\check{L}^+_{(m)A}$ and arbitrary functions $\Omega_1(\lambda,t), \dots, \Omega_{r_A}(\lambda,t)$ entire on λ. Let us pointed out that the evolution law of the scattering matrix of the type (4.10) was firstly considered in Ref. [15].

The transformations (4.3) with matrixes $B(\lambda,t)$ commuting with matrix Y given by (4.11) (i.e. for $B \subset g_{0(Y)}$) form an infinitedimensional group of Backlund-transformations for equations (4.7). At $\frac{\partial B_\alpha}{\partial t} = 0$ the transformations (4.3), as it follows from (2.7), does not change the evolution law (4.10) of the scattering matrix and therefore they are auto Backlund-transformations for equations (4.7): they transform solutions of the certain equation of the form (4.7) into the solutions of the same equation. Some concrete auto Backlund-transformations which have been found by the others techniques in Refs. [20,21] are particular cases of the general transformations (4.3). If $\frac{\partial B_\alpha}{\partial t} \neq 0$ then the transformations (4.3) are generalized Backlund transformations. Group B of the transformations (4.3) contains also as a subgroup an infinitedimensional symmetry group of the equations (4.7). In the infinitesimal form these symmetry transformations for regular matrix A are ($P \to P' = P + \delta P$)

$$\delta P(x,y,t) = \sum_{\alpha=1}^{N} \left(f_\alpha(\check{L}^+_A) H_\alpha P - f_\alpha(\hat{L}^+_A) P H_\alpha \right)$$

where $f_\alpha(\lambda)$ are arbitrary entire functions. Let us pointed out that group of Backlund transformations (4.3) and symmetry group are universal one, i.e. they are group of Backlund transformations and symmetry group for all equations of the form (4.7).

Let us attract now attention to the fact that one can obtain the equations (4.7) without using the transformations (4.3). Indeed, let the transformation T: $P \to P'$, $S \longrightarrow S'$ is the infinitesimal displaycement in time t: $P' = P + \varepsilon \frac{\partial P}{\partial t}$, $S' = S + \varepsilon \frac{dS}{dt}$, $\varepsilon \to 0$. From the relation (2.6) we obtain

$$\frac{d\hat{S}(\tilde{\lambda},\lambda,t)}{dt} = -\int_{-\infty}^{+\infty} d\mu\, \hat{S}(\tilde{\lambda},\mu) \int_{-\infty}^{+\infty} dx\,dy\, \check{F}_{\mu}^{+}(x,y) \frac{\partial P(x,y,t)}{\partial t} \hat{F}_{\lambda}^{+}(x,y). \tag{4.12}$$

Using (4.12) and the identity (2.8) one can show that

$$\frac{d\hat{S}(\tilde{\lambda},\lambda,t)}{dt} - \left(Y(\tilde{\lambda},t)\hat{S}(\tilde{\lambda},\lambda,t) - \hat{S}(\tilde{\lambda},\lambda,t) Y(\lambda,t) \right) =$$

$$= -\int_{-\infty}^{+\infty} d\mu\, \hat{S}(\tilde{\lambda},\mu,t) \int_{-\infty}^{+\infty} dx\,dy\, \check{F}_{\mu}^{+}(x,y) \Big(\frac{\partial P(x,y,t)}{\partial t} \hat{F}_{\lambda}^{+}(x,y) - \tag{4.13}$$

$$- Y\left(\frac{\partial}{\partial y},t\right) P(x,y,t) \hat{F}_{\lambda}^{+}(x,y) + P(x,y,t)\, Y\left(\frac{\partial}{\partial y},t\right) \hat{F}_{\lambda}^{+}(x,y) \Big)$$

where $Y(\lambda,t)$ is arbitrary matrix commuting with matrix A

$$\Big(Y = Y_{0}, \quad Y(\lambda,t) = \sum_{\alpha=1}^{r_A} H_{\alpha} \sum_{n=0}^{\infty} \omega_{\alpha n}(t) \lambda^{n} \Big).$$

Then taking into account the relations (3.6) and (3.8) (at $P' = P$) from (4.13) we finally obtain

$$\int_{-\infty}^{+\infty} d\mu \Big\{ \check{S}(\tilde{\lambda},\mu,t) \Big[\frac{d\hat{S}(\mu,\lambda,t)}{dt} - \left(Y(\mu,t)\hat{S}(\mu,\lambda,t) - \hat{S}(\mu,\lambda,t) Y(\lambda,t) \right) \Big] \Big\}_{F} =$$

$$= -\int_{-\infty}^{+\infty} dx\,dy\, \mathrm{tr} \Big\{ \overset{++}{\varphi}{}^{(F)}_{(0)F}(x,y;\tilde{\lambda},\lambda) \cdot \Big(\frac{\partial P(x,y,t)}{\partial t} - \tag{4.14}$$

$$- \sum_{\alpha=1}^{r_A} \left(\Omega_{\alpha}(\check{L}_{A}^{+},t) H_{\alpha} P - \Omega_{\alpha}(\hat{L}_{A}^{+},t) P H_{\alpha} \right) \Big) \Big\}.$$

From the equality (4.14) follows a close connection between the equations (4.7) and (4.10). In particular, if scattering matrix $\hat{S}(\tilde{\lambda},\lambda,t)$ satisfies the equation (4.10) then potential $P(x,y,t)$ satisfies the evolution equation (4.7) (if $\overset{++}{\varphi}{}^{(F)}_{(0)F}$ form a complete set).

Let us note also that the transformations (4.3) and equations (4.7) can be written in a form containing only one of the operators $\hat{\Lambda}^{+}_{(n)A}$, $\check{\Lambda}^{+}_{(n)A}$. For example, in the form

$$\sum_{\alpha=1}^{r_A} \sum_{n=0}^{\infty} b_{\alpha n}(t) \Big(\sum_{m=0}^{n} C_m^n \hat{\Lambda}^{+}_{(m)A} H_{\alpha} P'_{(n-m)} - \hat{\Lambda}^{+}_{(n)A} P H_{\alpha} \Big) = 0 \tag{4.15}$$

and

$$\frac{\partial P(x,y,t)}{\partial t} - \sum_{\alpha=1}^{r_A} \sum_{n=0}^{\infty} \omega_{\alpha n}(t) \Big(\sum_{m=0}^{n} C_m^n \hat{L}^{+}_{(m)A} H_{\alpha} P_{(n-m)} - \tag{4.16}$$

$$- \hat{L}^{+}_{(n)A} P H_{\alpha} \Big) = 0.$$

The equations (4.7) (or (4.16)) are just the nonlinear evolution equations in 1+2 dimensions (one time and two spatial dimensions) integrable by IST method with the help of the linear problem (1.1). Using the twodimensional version of IST method (see e.g. [1,13,15]) one can find, in principle, a broad class of exact solutions of the equations (4.7).

The class of equations (4.7) contains some well known nonlinear evolution equations in 1+2 dimensions. For example, at diagonal matrix A ($A_{ik} = a_i \delta_{ik}$, $a_i \neq a_k$, $i,k = 1,\dots,N$) and $Y_{ik}(\lambda) = \lambda \omega_i \delta_{ik}$, $i,k = 1,\dots,N$ where ω_i are some constants, the equation (4.7) is

$$\frac{\partial P_{ik}(x,y,t)}{\partial t} + \frac{\omega_i - \omega_k}{a_i - a_k} \cdot \frac{\partial P_{ik}}{\partial x} + \frac{a_k \omega_i - a_i \omega_k}{a_i - a_k} \cdot \frac{\partial P_{ik}}{\partial y} + \tag{4.17}$$

$$+ \sum_{\ell=1}^{N} \left(\frac{\omega_i - \omega_\ell}{a_i - a_\ell} - \frac{\omega_\ell - \omega_k}{a_\ell - a_k} \right) P_{i\ell} P_{\ell k} = 0. \quad (i,k = 1,\dots,N)$$

The system of equations (4.17) describing the twodimensional resonantly interacting waves has been studied (at N = 3) in Refs. [13,14,17].

As the second example we consider the case

$$A = \begin{pmatrix} I_N & 0 \\ 0 & -I_M \end{pmatrix}, \quad Y(\lambda) = 2i\lambda^2 A, \quad P = \begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix} \qquad (4.18)$$

where I_N and I_M are identical square matrixes of the order N and M, respectively, Q is N x M rectangular matrix and R is M x N rectangular matrix. In this case the equation (4.7) is the following system of matrix equations

$$i\frac{\partial Q}{\partial t} + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) Q - Q V_2 - V_1 Q = 0,$$

$$i\frac{\partial R}{\partial t} - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) R + R V_1 + V_2 R = 0,$$

$$V_1(x,y) = \int_{-\infty}^{x} dz\, \left((QR)' - (QR)''\right)(z, z-x+y), \qquad (4.19)$$

$$V_2(x,y) = \int_{-\infty}^{x} dz\, \left((RQ)' - (RQ)''\right)(z, x-z+y)$$

where $f'(x,y) = \frac{\partial f(x,y)}{\partial x}$, $f''(x,y) = \frac{\partial f(x,y)}{\partial y}$. At N = M = 1 the system of equations (4.19) firstly has been consider in Refs. [14,15].

At M = 1, arbitrary N and R = Q^+ system (4.19) reduces to 1+2 dimensional generalization of the N-component nonlinear Schrödinger equation ($Q = \begin{pmatrix} q_1 \\ \vdots \\ q_N \end{pmatrix}$)

$$i\frac{\partial q_k}{\partial t} + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) q_k - \sum_{n=1}^{N} \left(\delta_{kn} V_{(2)} + V_{(1)kn}\right) q_n = 0,$$
$$(k = 1, \dots, N)$$

$$V_{1\,kn}(x,y) = \int_{-\infty}^{x} dz \left((q_k q_n^*)' - (q_k q_n^*)''\right)(z, z-x+y),$$

$$V_{(2)}(x,y) = \int_{-\infty}^{x} dz \sum_{e=1}^{N} \left((q_e q_e^*)' - (q_e q_e^*)''\right)(z, x-z+y) \qquad (4.20)$$

At N = 1 see [14,15].

Let us note that in contrast to 1+1 dimensional differential equations integrable by the problem (1.0) the equations (4.7) are integro-differential one as a rule. The integro-differential equations (4.19), (4.20) can be also rewritten in the form of the systems of differential equations.

It is interesting to consider also the stationary equations (4.7), i.e. the equations (4.7) with $\frac{\partial P}{\partial t} = 0$. These equations

$$\sum_{\alpha=1}^{r_A} \left(\Omega_\alpha(\check{L}_A^+, t) H_\alpha P - \Omega_\alpha(\hat{L}_A^+, t) P H_\alpha\right) = 0$$

are the twodimensional equations which contain the independent variables x and y in more equal footing than 1+1 dimensional equations (e.g. the equations (4.7) at $\frac{\partial P}{\partial y} = 0$) contain the variables t and x. For example, at N = M = 1, $Q = R = U(x,y)$ the stationary equation (4.19) is equivalent to the following twodimensional system of equations for two scalar fields $U(x,y)$ and $\varphi(x,y)$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) U = U^3 + U\varphi,$$

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) \varphi = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^2 U^2$$

The system of nonstationary equations close to this system has been considered in Refs. [1,18].

At $\frac{\partial P}{\partial y} = 0$ the linear problem (1.1) is reduced to the onedimensional bundle (1.0) for the Fourier components of $\Psi(x,y,t)$ over variable y and the transformations (4.3) and equations (4.7) are reduced to the corresponding transformations and equations for bundle (1.0) [7-9].

V. Integrals of motion

Here for simplicity we consider the case of regular matrix A. In this case subalgebra g_0 is abelian one.

Let us note that in virtue of (4.10) the quantity $\hat{S}_0(\lambda,\lambda)$ is time independent:

$$\frac{d\hat{S}_0(\lambda,\lambda)}{dt} = 0 \qquad (5.1)$$

for any functions $\Omega_\alpha(\lambda,t)$. Therefore $S_0(\lambda,\lambda)$ at any λ are integrals of motion. Expanding (analogously to 1+1 dimensional case [1,2]) the quantity $\ln \hat{S}_0(\lambda,\lambda)$ in the asymptotic series on λ^{-1}: $\ln \hat{S}_0(\lambda,\lambda) = \sum_{n=1}^{\infty} \lambda^{-n} C^{(n)}$ we obtain the counting set of the integrals of motion $C^{(n)}$ (n = 1,2,...) for equations (4.7). Analogously to 1+1 dimensional case the integrals of motion $C^{(n)}$ can be written as a functionals over potential $P(x,y,t)$. Let us represent for this purpose the matrix-solution $\hat{F}^+_\lambda(x,y,t)$ in the form

$$\hat{F}^+_\lambda(x,y,t) = R_\lambda(x,y,t)\, E_\lambda(x,y)\, e^{\chi_\lambda(x,y,t)} \qquad (5.2)$$

where $E_\lambda(x,y) = \exp \lambda(y - Ax)$, $(\chi_\lambda)_0 = \chi$ and $(R_\lambda)_0 = 1$. Passing in (5.2) to the limit $x \to -\infty$ and taking the projection on the subspace g_0 we obtain

$$\ln \hat{S}_0(\lambda,\lambda) = \ln\left(\int_{-\infty}^{+\infty} dy\, e^{\chi_\lambda(-\infty,y)}\right) \qquad (5.3)$$

Thus the integrals of motion $C^{(n)}$ are the coefficients in the asymptotic expansion on λ^{-1} of the right-hand side of the equality (5.3). These coefficients are connected in obvious way with the coefficients $\chi_{(n)}(y)$ of the asymptotic expansion on λ^{-1} of the quantity $\chi_\lambda(-\infty,y)$ $\left(\chi_\lambda(-\infty,y) = \sum_{n=1}^{\infty} \lambda^{-n} \chi_{(n)}(y)\right)$.

As a result

$$C^{(1)} = \int_{-\infty}^{+\infty} dy\, \chi_{(1)}(y),$$

$$C^{(2)} = \int_{-\infty}^{+\infty} dy\, \left(\chi_{(2)}(y) + \frac{1}{2}\chi^2_{(1)}(y)\right) - \frac{1}{2}\left(\int_{-\infty}^{+\infty} dy\, \chi_{(1)}(y)\right)^2$$

and so on.

The quantities $\chi_{(n)}(y)$ are found from the recursion relations which analogously to 1+1 dimensional case are obtained by substituting $\hat{F}^+_\lambda(x,y)$ in the form (5.2) into linear problem (1.1). They are of the form

$$\chi_{(n)}(y) = -\mathcal{Y}^{-}(PR^{(n)})_0 \qquad (n=1,2,\ldots) \qquad (5.4)$$

where $R^{(n)}$ are calculated by the recursion relations ($R_\lambda(x,y) = 1 + \sum_{n=1}^{\infty} \lambda^{-n} R^{n}(x,y)$)

$$R^{(1)} = P_A,$$

$$R^{(n+1)} = -\left(\frac{\partial}{\partial x} + A\frac{\partial}{\partial y}\right) R^{(n)}_A - \sum_{q=1}^{n-1} R^{(q)}_A \frac{\partial}{\partial x}\mathcal{Y}^{-}(PR^{(n-q)})_0 - A\sum_{q=1}^{n-1} R^{(q)}_A \frac{\partial}{\partial y}\mathcal{Y}^{-}(PR^{(n-q)})_0 + (PR^{(n)})_{F=0} \qquad (5.5)$$

$$(n = 1, 2, \ldots)$$

Let us emphasize that the integrals of motion $C^{(n)}$ are universal one. Indeed in their calculation we use only the fact of time independence of $\hat{S}_0(\lambda,\lambda)$ and spectral problem (1.1) but not the equations (4.7). Therefore $C^{(n)}$ are integrals of motion for any equations of the form (4.7).

In the particular case of diagonal regular matrix A and N = 3 the procedure for calculation of the integrals of motion in 1+2 dimensions described in this section is close to those given earlier in Refs. [17,18,22].

VI. Reduction problem

Similar to 1+1 dimensions case the reduction problem for general equations (4.7), i.e. the problem of effective decreasing of the number of the independent fields in these equations is an important one.

In the 1+1 dimensions case A.V.Mikhailov [23,24] proposed very interesting approach to the reduction problem. This approach is based on the introduction of the notion of the reduction group, in other words, the group of the form-invariance of potential. In the framework of AKNS-approach the reduction problem leads also to the problem of enumeration of those functions $\Omega(\lambda, t)$ for which the integrable equations admit certain reduction. In the 1+1 dimensions case and bundle (1.0) this problem was solved in Ref. [25]. A close approach for another spectral problem was proposed in Refs. [26,27].

Here we consider the reduction problem for the equations (4.7) integrable by the problem (1.1). Let us consider for definiteness Z_N reduction. In 1+1 dimensions case see [23,24,25]. Z_N - reduction is generated by the constraints

$$GA = qAG, \quad GP(x,y,t) = P(x,y,t)G \qquad (6.1)$$

where $G_{ik} = \delta_{i,k-1}$ $(i,k=1,\dots,N)$, $G_{N1}=1$ and $q = \exp\frac{2\pi i}{N}$. Under Z_N - reduction the potential P have only N-1 independent variables and $A_{ik} = q^{i-1}\delta_{ik}$ [23-25]. Let $\hat\varphi_\lambda(x,y)$ is some solution of the problem (1.1). Let us consider the quantity $\psi'(x,y) = G\hat\varphi_\lambda(x,y)$. At $|x| \to \infty$ it satisfies the equation $\frac{\partial\psi'}{\partial x} + qA\frac{\partial\psi'}{\partial y} = 0$ and therefore at $|x| \to \infty$ it can be represented as follows

$$G\hat\varphi_\lambda(x,y) = \int_{-\infty}^{+\infty} d\mu\, \hat\varphi_\mu(x, q^{-1}y)\, T(\mu,\lambda) \qquad (6.2)$$

where $T(\mu,\lambda)$ is some matrix. Using (2.2) from (6.2) we obtain

$$T(\mu,\lambda) = \int_{-\infty}^{+\infty} dy\, \check\varphi_\mu(x,y)\, G\hat\varphi_\lambda(x, qy)\Big|_{|x|\to\infty}. \qquad (6.3)$$

Putting in (6.3), for example, $\hat\varphi_\lambda = \hat F^+_\lambda$ and $x \to +\infty$ one get

$$T(\mu,\lambda) = \delta(\mu - q\lambda)\,G. \qquad (6.4)$$

As a result the relation (6.2) take the form

$$G\hat\varphi_\lambda(x,y)\Big|_{|x|\to\infty} = \hat\varphi_{q\lambda}(x, q^{-1}y)\,G\Big|_{|x|\to\infty}. \qquad (6.5)$$

Further, since the relation (6.2) is valid at $|x| \to \infty$ both for F^+ and F^- we obtain the following equation for the scattering matrix

$$\int_{-\infty}^{+\infty} d\mu\, \hat S(\tilde\lambda, \mu)\, T(\mu,\lambda) = \int_{-\infty}^{+\infty} d\mu\, T(\tilde\lambda,\mu)\, \hat S(\mu,\lambda). \qquad (6.6)$$

Taking into account (6.4) we have

$$G\hat S(\tilde\lambda, \lambda, t)\,G^{-1} = S(q\tilde\lambda, q\lambda, t). \qquad (6.7)$$

Further, demanding the consistence of the constraint (6.7) with the equation (4.10) we obtain

$$GY(\lambda,t)\,G^{-1} = Y(q\lambda, t). \qquad (6.8)$$

Since $Y(\lambda,t) = \sum_{\alpha=1}^{r_A} \Omega_\alpha(\lambda,t) H_\alpha$ from (6.8) we find that $\Omega_\alpha(\lambda,t) = \Omega_1(q^{\alpha-1}\lambda, t)$ $(\alpha=1,\dots,N)$ i.e. $\Omega_\alpha(\lambda,t) = \sum_{n=1}^{N-1} q^{(\alpha-1)n}\lambda^n\,\Omega_n(\lambda^N, t)$ where $\Omega_n(\lambda^N, t)$ are arbitrary functions entire on λ^N. As a result

$$Y(\lambda,t) = \sum_{n=1}^{N-1} \lambda^n\,\Omega_n(\lambda^N, t)\,A^n. \qquad (6.9)$$

The expression (6.9) give us the general form of $Y(\lambda,t)$ for which the equations (4.7) admit a nontrivial Z_N - reduction on (6.1). Indeed for such $Y(\lambda,t)$ the relation (6.7) is consistent with the equation (4.10) and as a result, as it follows from the equality (4.14), the constraint (6.1) is consistent with the equation (4.7).

In the general case dealing with the reduction problem one can acts in the same way as in the example considered above:

1) firstly we find the reduction group, i.e. find the constraints on matrix A and potential P,

2) calculate matrix $T(\mu,\lambda)$ using (6.3),

3) obtain the constraint for $Y(\lambda,t)$ analogous to (6.8) and

4) solving this constraint one find the general form of the functions $\Omega_\alpha(\lambda,t)$ for which the equations (4.7) admit the given reduction.

Let us consider as an illustrations some other reductions. So the general equations (4.7) admit the reduction $P^T(x,y,t) = -P(x,y,t)$ at $A^T = A$ and arbitrary odd functions $\Omega_\alpha(\lambda,t)$.

Similar to 1+1 dimensions case [28] so called $\widetilde{Z}_N$ - reduction is important. This is the reduction

$$A = \begin{pmatrix} 1 & & & 0 \\ & q & & \\ & & \ddots & \\ 0 & & & q^{N-1} \end{pmatrix}, \quad P_{\widetilde{Z}_N} = \begin{pmatrix} 0 & Q_{N-2} & Q_{N-1} & \dots & Q_2 & Q_1 & Q_0 \\ 1 & 0 & Q_{N-2} & \dots & \dots & Q_2 & Q_1 \\ 0 & 1+q & 0 & \dots & \dots & \dots & Q_2 \\ 0 & 0 & 1+q+q^2 & \ddots & & & \vdots \\ \vdots & & & \ddots & \ddots & & \vdots \\ \vdots & & & & \ddots & \ddots & Q_{N-2} \\ 0 & 0 & \dots & \dots & \dots & 1+q+\dots+q^{N-1} & 0 \end{pmatrix} \tag{6.10}$$

where $q = \exp\frac{2\pi i}{N}$. One can to show that the equations (4.7) admit $\widetilde{Z}_N$-reduction (6.10) at $Y(\lambda,t) = \sum_{n=1}^{N-1} \lambda^n \Omega_n(\lambda^N,t) A^n$, i.e. for the same class of functions Ω_α as for Z_N -reduction (6.1).

It is not difficult to prove that the linear problem (1.1) under $\widetilde{Z}_N$ -reduction is equivalent to the following twodimensional problem ($\Psi^T = (\Psi_1,\dots,\Psi_N)$)

$$\sum_{k=0}^{N} V_k(x,y)\frac{\partial^k}{\partial x^k}\Psi_N - \frac{\partial^N}{\partial y^N}\Psi_N = 0 \tag{6.11}$$

where $V_N = 1$, $V_{N-1} = 0$. Coefficients $V_0(x,y),\dots, V_{N-2}(x,y)$ are simply expressed through $Q_0,\dots, Q_{N-2}$. For example, at N = 2 $(q=-1)$ $V_0(x,y) = -Q_0(x,y)$. The corresponding spectral problem is

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)\Psi_2 + V_0(x,y)\Psi_2 = 0$$

and it has been considered in Ref. [29]. At N = 3

$$V_0 = -(1+q)\left(\frac{\partial Q_1}{\partial x} + \frac{\partial Q_1}{\partial y} + Q_0\right), \quad V_1 = -(2+q)\,Q_1$$

or

$$Q_0 = -\frac{1}{1+q}V_0 + \frac{1}{2+q}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)V_1, \quad Q_1 = -\frac{1}{2+q}V_1.$$

Spectral problem (6.11) is one of the possible twodimensional generalizations of the well known Gelfand-Dikij spectral problem $\sum_{k=0}^{N} V_k(x)\frac{\partial^k}{\partial x^k}\chi - \lambda^N\chi = 0$. Another twodimensional generalization of Gelfand-Dikij problem is $\sum_{k=0}^{N} V_k(x,y)\frac{\partial^k}{\partial x^k}\chi + \alpha\frac{\partial\chi}{\partial y} = 0$ and it can be obtained as a special reduction of the problem (1.1) with degenerated matrix A. For example, the second order problem $\frac{\partial\Psi}{\partial x} + \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}\frac{\partial\Psi}{\partial y} = \begin{pmatrix} 0 & u \\ 1 & 0 \end{pmatrix}\Psi$ is equivalent to the scalar problem $\alpha\frac{\partial\Psi_2}{\partial y} + \frac{\partial^2\Psi_2}{\partial x^2} - u(x,y,t)\Psi_2 = 0$ which is used for integration of Kadomtsev-Petviashvili equation [13,30].

Let us consider the families of the equations (4.7) under Z_2 and $\widetilde{Z}_2$ -reductions (N=2). From the obtained results follows that the equations (4.7) admit these reductions at $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $Y = \lambda\Omega(\lambda^2)A$ where $\Omega(\lambda^2)$ is arbitrary entire function on λ^2. At Z_2 reduction $P = \begin{pmatrix} 0 & q \\ q & 0 \end{pmatrix}$, for $\widetilde{Z}_2$ -reduction $P = \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix}$. At $\Omega(\lambda^2) = -2^{2n}(\lambda^2)^n$ $(n = 1,2,\dots)$ and $\widetilde{Z}_2$ -reduction a family of the equations (4.7) (N=2) is a generalization of the well known KdV-family to 1+2 dimensions. The simplest (n=1) from these equations is 1+2 dimensions generalization of KdV-equation considered in Ref. [29]. At $\Omega(\lambda^2) = -2^{2n}(\lambda^2)^n$ (n=1,2,...) and Z_2 -reduction the family of the equations (4.7) (N=2) is 1+2 dimensions generalization of the mKdV-family. The simplest (n=1) of these equations is generalization of modified KdV-equation on 1+2 dimensions.

Let us in conclusion attract attention to one important circumstance. All the formulas which we use above in the analysis of the reductions contain only asymptotics of the solutions of the problem (1.1) at $|x| \to \infty$. We also does not demand that the problem (1.1) is invariant under the transformation $\hat{\varphi}_\lambda(x,y) \to \hat{\varphi}'_\lambda(x,y) = G\hat{\varphi}_\lambda(x,y)$ i.e. we does not demant that $G\hat{\varphi}_\lambda(x,y)$ is the solution of the problem (1.1). In this point there is an important difference between the case of the twodimensional problem (1.1) and the case of the onedimensional bundle (1.0). While for onedimensional bundle the reduction group, i.e. the group of the form-invariance of the potential, is in the same time the symmetry group of the bundle (1.0), for twodimensional problem the reduction group (group of form-invariance of potential $P(x,y,t)$) is not a symmetry group of the problem (1.1).

Of course, one may demand in the twodimensional case that the reduction group will be the symmetry group of the problem (1.1) too. However in this case, as it is not difficult to see, we must demand that the variable y is transformed in a non-trivial way. For example, for Z_N reduction we must demand $y \to y' = q^{-1}y$ and instead of (6.1) the potential $P(x,y,t)$ should satisfies the constraint $GP(x,y,t)G^{-1} = P(x, q^{-1}y, t)$ for the whole ranges of the variables x and y. The reasonable interpretation of such constraints is not clear.

VII. Conclusion

The results of the present paper, analogously to 1+1 dimensions [8, 25], can be generalized to the problem (1.1) with Z_2-grading (potential contains both commuting and anticommuting fields), to the case when $\lim_{R=\sqrt{x^2+y^2}\to\infty} P(x,y,t) \neq 0$ and to others spectral problems. Hamiltonian and group-theoretical structure of the evolution equations (4.7) will be considered elsewhere.

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