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A b s t r a c t

It is proposed a generalization of the method of group analysis of differential equations to the symmetry groups more general than Lie-Backlund groups, in particular, to the non-local groups of transformations.

1. INTRODUCTION

The invariance of the differential equations of mathematical physics under some group of transformations is one of most important properties of these equations. S.Lie proposed the systematic method to search for the groups of symmetry of differential equations which was developed later by L.V.Ovsjannikov [1].

In recent years it turned out that differential equations can have the symmetry groups which cannot be described within the framework of the Lie-Ovsjannikov method [2] . In refs. [3-5] a wider scheme which includes new groups of symmetry was constructed. The Lie-Backlund groups considered in these papers are characterized by the fact that the corresponding transformation laws contain the arbitrary-order derivatives. All the local conservation laws may appear to be described in the framework of these symmetries.

However together with the local conservation laws it makes sense to consider the non-local ones which can carry an essential information on differential equations. One can find the examples of non-local conservation laws constructed in the study of some differential equations in papers [6,7] . The groups of symmetry responsible for an appearance of the non-local conserved quantities cannot be described in the Anderson-Ibragimov scheme.

In the present paper it is suggested a generalization of the scheme of the group theoretic analysis of differential equations. This allows us to describe the groups of symmetry of differential equations which are more general than the Lie-Backlund groups. A major attention is here paid to the groups of non-local transformations.

The formalism proposed is applicable not only to differential but also to integro-differential equations.

In section II the Lie-Ovsjannikov scheme is reviewed briefly. Unlike the standard approach (see, e.g.,[1]), we shall use thorough the infinitesimal form of description. It is usually introduced (see ref. [1], pp.46-58), first of all, the notion of prolongation of a space, then that of prolongation of mappings, and finally, the notion of prolongation of transformations. Here we keep to the inverse sequence which is most convenient and transparent.

The infinitesimal form has been employed in the book [8] as well. But the advantages of this approach are used, in our opinion, in the present paper more completely. In particular, it is derived the formula (3) greatly simplifying the calculations and proofs.

In section III the Lie-Ovsjannikov scheme is modified in such a form wherein the arbitrary functions are independent variables.

In section IV this scheme is generalized for the case of arbitrary integro-differential transformations.

In section V some questions of the symmetry of differential equations and also the relationship between symmetry groups and conservation laws are discussed.

II. The Lie-Ovsjannikov scheme in the infinitesimal form

Just as in ref. [1], the operator $\hat{X}$ will be called the infinitesimal operator of the symmetry group (or, briefly, the symmetry operator) of the differential equation $\omega = 0$ if $\hat{X}\omega = 0$ *. Two possibilities have been previously considered:

* We use the notation and the terminology of refs. [1, 3-5].

1) $\hat{X}\omega = 0$ in virtue of the initial differential equation - the Lie-Ovsjannikov symmetries. This class also involves the group of Lie tangent transformations (see ref. [1]).

2) $\hat{X}\omega = 0$ in virtue of differential equation itselt and its any differential consequencies - the Lie-Backlund groups [2-5].

Let us remind briefly the Lie-Ovsjannikov scheme. Let $\{x^i\}$ $(i = 1,\dots,N)$ be a set of independent variables, and $\{u^\kappa\}$ $(\kappa = 1,\dots,M)$ be dependent variables. One writes down the transformation law in the infinitesimal form:

$$\begin{aligned} x^i &\to \tilde{x}^i = x^i + \epsilon\, \xi^i(x,u) \\ u^\kappa &\to \tilde{u}^\kappa = u^\kappa + \epsilon\, \eta^\kappa(x,u) \end{aligned} \qquad (1)$$

Here ϵ is the group parameter, $\epsilon\xi^i$ and $\epsilon\eta^\kappa$ are the variations of the corresponding variables.

At the first stage the arbitrary scalar functions $u^\kappa = u^\kappa(x)$ are considered as the functions given in Banach-space $\mathcal{X}$ with the coordinates $\{x^i\}$. For convenience, the operator of partial differentiation will be denoted by $\mathfrak{D}_i$. In other words:

$$\mathfrak{D}_i u^\kappa = u^\kappa_i \;, \quad \mathfrak{D}_j u^\kappa_i = u^\kappa_{ij} \qquad \text{and so on.} \qquad (2)$$

The transformation law for partial derivatives $u^\kappa_{i_1\dots i_n}$ is, of course, determined from eqs.(1) and (2). $\tilde{\mathfrak{D}}_i$ stands for the operator of partial differentiation with respect to $\tilde{x}^i$. It is natural to demand that the relations (2) be invariant under transformations (1), i.e. that, for example, $\mathfrak{D}_i u^\kappa = u^\kappa_i$ is transformed into $\tilde{\mathfrak{D}}_i \tilde{u}^\kappa = \tilde{u}^\kappa_i$. Using the well known chain rule $\tilde{\mathfrak{D}}_i = (\tilde{\mathfrak{D}}_i x^j)\mathfrak{D}_j$ twice, one can expresses $\tilde{\mathfrak{D}}_i$ via $\mathfrak{D}_i$ with an accuracy up to the first order on ϵ.

$$\tilde{\mathfrak{D}}_i = (\tilde{\mathfrak{D}}_i x^j)\mathfrak{D}_j = \tilde{\mathfrak{D}}_i(\tilde{x}^j - \epsilon\xi^j)\mathfrak{D}_j = \mathfrak{D}_i - \epsilon(\tilde{\mathfrak{D}}_i \xi^j)\mathfrak{D}_j =$$

$$= \mathcal{D}_i - \epsilon(\mathcal{D}_\ell \xi^j)(\tilde{\mathcal{D}}_i x^\ell)\mathcal{D}_j = \mathcal{D}_i - \epsilon(\mathcal{D}_\ell \xi^j)(\tilde{\mathcal{D}}_i(\tilde{x}^\ell - \epsilon \xi^\ell))\mathcal{D}_j =$$

$$= \mathcal{D}_i - \epsilon(\mathcal{D}_i \xi^j)\mathcal{D}_j + O(\epsilon^2)$$

For convenience, let us introduce the following notation $\mathcal{D}_i \xi^j \equiv \xi^j_i$. Note however that the expressions $\mathcal{D}_i \eta^\kappa$ and η^κ_i have a different load. So,

$$\tilde{\mathcal{D}}_i = \mathcal{D}_i - \epsilon\, \xi^j_i\, \mathcal{D}_j \tag{3}$$

And applying eq.(3) again, one gets:

$$\tilde{\mathcal{D}}_{i_1}\tilde{\mathcal{D}}_{i_2}\cdots\tilde{\mathcal{D}}_{i_n} = \mathcal{D}_{i_1}\mathcal{D}_{i_2}\cdots\mathcal{D}_{i_n} - \epsilon\, \xi^\ell_{i_1\dots i_n}\mathcal{D}_\ell - \epsilon \sum_{\nu=1}^{n-1}\sum_{(\nu)} \xi^\ell_{i_1\dots i_\nu}\mathcal{D}_{i_{\nu+1}}\cdots\mathcal{D}_{i_n}\mathcal{D}_\ell \tag{4}$$

where $\sum_{(\nu)}$ denotes the summation over all possible ν-elements samples from n. The samples $i_1 \cdots i_{n_1}$ and $j_1 \cdots j_{n_2}$ are considered to be different only in the case when $n_1 \neq n_2$, or when among $i_1 \cdots i_n$ there is at least one number not equal to any of $j_1 \cdots j_n$.

To find the prolongation of transformations (1) on the variables $u^\kappa_i, u^\kappa_{ij}, \dots$ it is necessary to express $\eta^\kappa_i, \eta^\kappa_{ij}, \dots$ in $\tilde{u}^\kappa_i = u^\kappa_i + \epsilon\eta^\kappa_i$, $\tilde{u}^\kappa_{ij} = u^\kappa_{ij} + \epsilon\eta^\kappa_{ij}$ and so on via ξ^i and η^κ.

The prolongation of transformations (1) on the variables are obtained from eqs.(1) and (3):

$$\tilde{\mathcal{D}}_i \tilde{u}^\kappa \equiv \tilde{u}^\kappa_i = (\mathcal{D}_i - \epsilon\xi^j_i\mathcal{D}_j)(u^\kappa + \epsilon\eta^\kappa) =$$

$$= u^\kappa_i + \epsilon(\mathcal{D}_i\eta^\kappa - u^\kappa_j\xi^j_i) - O(\epsilon^2)$$

Whence:

$$\eta^\kappa_i = \mathcal{D}_i\eta^\kappa - u^\kappa_j\xi^j_i \tag{5}$$

This is the known formula of the prolongation of transformations (1) on the <u>variables</u> u^κ_i *[1]. The prolongation of transformations (1) on the variables $u^\kappa_{i_1\dots i_n}$ is found in an analogous way, using eq.(1).

The second method for finding the prolongation of transformations on u^κ_{ij} consists in applying eq.(3) to eq.(5) and so on. It is more convenient to derive the prolongation formula by means of eq.(4) in a sense that in this case the intermediate differentiations have been already performed.

After deriving the formulae of the prolongations of transformations (1) on the variables $u^\kappa_{i_1\dots i_n}$ the notion of the prolongation of mappings is introduced: from the mapping $u: \mathcal{X} \to \mathcal{Y}$ of a space $\mathcal{X}$ in $\mathcal{Y}$, which gives by the formula $y^\kappa = u^\kappa(x), (y^\kappa \in \mathcal{Y})$, it is constructed the prolongation of this mapping to the mappings $\underset{1}{u}^\kappa: \mathcal{X} \to \underset{1}{\mathcal{Y}}$; $\underset{2}{u}^\kappa: \mathcal{X} \to \underset{2}{\mathcal{Y}}$ and so on. Correspondingly, the latter are given by formulae [1] : $\underset{1}{y}^\kappa = \underset{1}{\partial} u^\kappa(x)$; $\underset{2}{y}^\kappa = \underset{2}{\partial} u^\kappa(x)$, ... and so on. Here $\underset{i}{y}^\kappa \in \underset{i}{\mathcal{Y}}$; $\underset{i}{\partial}$ are the partial derivatives of the i-th order.

At the third stage the prolonged space is constructed:

$$\mathcal{X} \times \mathcal{Y} \times \underset{1}{\mathcal{Y}} \times \underset{2}{\mathcal{Y}} \times \cdots$$

All these constructions are required to have a possibility to treat the symmetry group of a differential equation as a group of transformations of the coordinates of the corresponding prolonged space. The differential equation itself is understood as an equation giving some submanifold in the prolonged space.

* The coordinates corresponding to u^κ_i are not yet introduced by us.

As easily see, the transformations of the coordinates of prolonged space form a group iff the transformation laws of the coordinates repeat those of their corresponding variables:

$$\begin{aligned} x^i &\to \tilde{x}^i = x^i + \epsilon\, \xi^i(x,y) \\ y^k &\to \tilde{y}^k = y^k + \epsilon\, \eta^k(x,y) \end{aligned} \qquad (6)$$

The tranaformations laws of the coordinates $\underset{1}{y}^k, \underset{2}{y}^k, \ldots$ repeat the transformation laws of $\underset{1}{u}^k, \underset{2}{u}^k, \ldots$. It will become obvious if we prove the formula analogous to formula (3) but for the total-differentiation operators being of the form:

$$D_i = \frac{\partial}{\partial x^i} + y^k_i \frac{\partial}{\partial y^k} + y^k_{ji} \frac{\partial}{\partial y^k_j} + \cdots$$

Since the coordinates $x^i, y^k, \underset{1}{y}^k, \ldots$ are independent, the ordinary partial differentiation, for example, of y^k with respect to x^i yields zero: $\frac{\partial y^k}{\partial x^i} = 0$. In this scheme the operator of total differentiation possesses all the properties of the partial differentiation in a usual sense.

If one introduces the notation:

$$\tilde{D}_i = \frac{\partial}{\partial \tilde{x}^i} + \tilde{y}^k_i \frac{\partial}{\partial \tilde{y}^k} + \tilde{y}^k_{ji} \frac{\partial}{\partial \tilde{y}^k_j} + \cdots$$

then it is possible to show that all the calculations leading to formulae (3) and (4) and the formulae themselves occur for the operators of total differentiation as well.* Best of all, this can be done as follows. Let us assume that formula (3) is also valid for the operators of total differentiation. As before, we obtain the prolongations of eq.(6) on $y^k_i, \ldots$, i.e. the analogue of formula (5). Knowing the prolongation of eq.(6) on $y^k_i, y^k_{ij}, \ldots$ it is easy to express $\tilde{D}_i$ via D_i , what leads to formula (3).

* For this reason we denote them by the same symbol.

Let us carry out these short calculations only for the case of "cut-off" operator $\tilde{D}_i = \frac{\partial}{\partial \tilde{x}^i} + \tilde{y}^k_i \frac{\partial}{\partial \tilde{y}^k}$

$$\tilde{D}_i = \left(\frac{\partial}{\partial x^i} - \epsilon\, \xi^j_i \frac{\partial}{\partial x^j}\right) + \left(y^k_i + \epsilon \frac{\partial \eta^k}{\partial x^i} - \epsilon\, u^k_j \xi^j_i\right) \frac{\partial y^\ell}{\partial \tilde{y}^k} \frac{\partial}{\partial y^\ell} =$$

$$= \left(\frac{\partial}{\partial x^i} - \epsilon\, \xi^j_i \frac{\partial}{\partial x^j}\right) + \left(y^k_i \frac{\partial}{\partial y^k} - \epsilon\, \xi^j_i y^k_j \frac{\partial}{\partial y^k}\right) -$$

$$- \epsilon\, y^k_i \frac{\partial \eta^\ell}{\partial y^k} \frac{\partial}{\partial y^\ell} + \epsilon\, y^\ell_i \frac{\partial \eta^k}{\partial y^\ell} \frac{\partial}{\partial y^k} = D_i - \epsilon\, \xi^j_i D_j$$

Thereby we satisfy ourselves that our assumptions true.

The differential equation is given in a prolonged space by the equation:

$$\omega(x, y, \underset{1}{y}, \ldots) = 0$$

The mapping $\mathcal{X} \to \mathcal{Y}$ is called the solution of a differential equation if

$$\omega(x, u(x), \underset{1}{u}(x), \ldots) = 0$$

for all $x \in \mathcal{X}$ [1].

It is seen from formula (5) that if ξ^i and (or) η^k depend on $\underset{1}{y}^k$, then η^k_i will in general case depend on $\underset{2}{y}^k$, then η^k_{ij} will depend on $\underset{3}{y}^k$ and so forth. In order that a group of transformations act in a finite-dimensional space, there exist two possibilities:

a) ξ^i and η^k depend only on x^i and y^k – the group of point transformations (and their prolongations);

b) ξ^i and η^k depend also on $\underset{1}{y}^k$ but x^i and y^k are transformed by the related laws (to exclude the dependence η^k_i on $\underset{2}{y}^k$) – the groups of Lie tangent transformations (and their prolongations).

The last case is possible only at $M=1$, at $M>1$ the dependence ξ^i and (or) η^κ on $\underset{1}{y}^\kappa$ leads us with necessity to the Lie-Backlund groups [3].

In this case formulae (6) take the more general form:

$$\tilde{x}^i = x^i + \epsilon\, \xi^i(x, y, \underset{1}{y}, \dots)$$
$$\tilde{y}^\kappa = y^\kappa + \epsilon\, \eta^\kappa(x, y, \underset{1}{y}, \dots) \tag{7}$$

III. Modification of the Lie-Ovsjannikov scheme

In the Lie-Ovsjannikov scheme $x^i, y^\kappa, \underset{1}{y}^\kappa_i, \dots$ are considered to be independent, but for an arbitrary function $f: \frac{\partial f(y^\kappa)}{\partial y^\kappa} \neq 0$. One can however modify this scheme so that to each arbitrary function of $x^i, u^\kappa, u^\kappa_i, \dots$ its own coordinate in the appropriate prolonged space is put into correspondence.*

With the known transformation law (7) and its prolongations on $\underset{1}{y}^\kappa, \underset{2}{y}^\kappa, \dots$ the prolongation on arbitrary functions is obtainable, by expansing in a series the transformed function and by neglecting the terms of the second order and higher on ϵ.

The operator of total differentiation is then written in the form:

$$\mathcal{D}_i = \int df\, f_i\, \partial_f (\quad) \tag{8}$$

where $f_i = \frac{\partial f}{\partial x^i}$, and $\partial_f = \frac{\partial}{\partial f}$. The brackets denote the operator character of this expression. Various functions are independent in a sense that $\frac{\partial f}{\partial \varphi} = \delta(f - \varphi)$ where $\delta(f-\varphi)$ is the Dirac delta-function.

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* Of course, the coordinates for f and $\alpha f + \beta$ are not considered to be independent (α, β are the numbers).

If in the Lie-Ovsjannikov scheme the operator of symmetry is written in the form:

$$\hat{X} = \xi^i \frac{\partial}{\partial x^i} + \eta^\kappa \frac{\partial}{\partial y^\kappa} + \eta^\kappa_i \frac{\partial}{\partial \underset{1}{y}^\kappa_i} + \dots \equiv \xi^{z^i} \frac{\partial}{\partial z^i}$$

where the notation $z^i = \{x^1, \dots, x^N; y^1 \dots y^M; \underset{1}{y}^1_1 \dots \underset{1}{y}^M_1; \underset{2}{y}^1, \dots\}$ is introduced, but now this operator is represented in the form:

$$\hat{X} = \int df\, \xi^f\, \partial_f (\quad)$$

Restricting the set of functions f by the standard basis z^i, we shall apparently return to the scheme of the papers [1,3].

In the approach proposed one may consider the left-hand side of the differential equation itself to be a new variable. Then transformations (6) induce the transformations and the task of searching for a symmetry group is formulated as follows:

The group of symmetry is seeked for which:

1) $\xi^\omega = f_0\, \omega$ (Lie-Ovsjannikov scheme), or

2) $\xi^\omega = f_0\, \omega + \sum_{i=1}^{N} f_i\, \mathcal{D}_i\, \omega + \sum_{i,j=1}^{N} f_{ij}\, \mathcal{D}_i \mathcal{D}_j\, \omega + \cdots$

(Anderson-Ibragimov scheme). Here $f_0, f_i, f_{ij}, \dots$ are the arbitrary functions of z^i, numerated by several indexes, $\mathcal{D}_i$ is the operator of total differentiation.

IV. Integro-differential prolongations

Besides two possibilities studied in the preceding section there is, as easily see, the third one: $\hat{X}\omega = 0$ due to the integro-differential consequencies of an equation, in other words:

$$\xi^{\omega} = \dots + \sum_{i=1}^{N} f_{-i} \int \psi_i \, \omega \, dx^i + f_0 \omega + \sum_{i=1}^{N} f_i \, \mathcal{D}_i \omega + \cdots$$

Here $\dots f_{-i}, \psi_i, f_0, f_i, \dots$ are the arbitrary functions of z^i.

Let us construct a scheme of the group theoretical analysis of differential equations, generating the scheme of works [1], [3-5] and taking account of this third possibility.

One has to modify the Lie-Ovsjannikov scheme in the form wherein to each arbitrary function its own coordinate is put into correspondence. The reason for this is that under consideration of integral prolongations the necessity arises to find prolongations of transformations (7) to the integrals of arbitrary functions. Together with introduction of the coordinate for $\int u^k dx^i$ it is required to introduce the corrdinates for $\int (u^k)^2 dx^i$; $\int u^k \underset{2}{u}^k dx^i$ as well, and so on, i.e. for the variables of general form $\int f(x, u, \underset{1}{u}, \dots) \, dx^i$.

In the case of Lie-Backlund groups, due to properties of the operator of symmetry as a differential operator it is possible to confine oneself to the basis $\{z^i\}$ since differentiating the arbitrary function of z^i, one obtains again the function of z^i, i.e. the basis $\{z^i\}$ is complete.

As readily see, in the case of integral prolongations the basis $\{z^i, \int z^i dx^j\}$ is not complete since, for instance, $\int (z^i)^2 dx^k$ is no longer expanded in this basis. Therefore, the action of the symmetry operator on the last function will not be determined. The way out of a situation is just to introduce the coordinates corresponding to the integrals of arbitrary functions.

We get the formulae analogous to (3) for integral operators. Let us consider now the integral $\int \Phi \, d\tilde{x}^i$ where Φ is the

arbitrary function. Under infinitesimal transformation $x^i \to \tilde{x}^i = x^i + \epsilon \xi^i$ and hence:

$$\int d\tilde{x}^i (\Phi) = \int \{ dx^i (\Phi) + \epsilon \, \xi^i_j (\Phi) \, dx^j \}$$

Omitting the arbitrary function Φ, one gets the analogue of formula (3)*:

$$\int d\tilde{x}^i (\ ) = \int \{ dx^i (\ ) + \epsilon \, \xi^i_j \, dx^j (\ ) \} \quad (9)$$

The integral prolongations can be found by formula (9). However we shall obtain them directly from the relation $\tilde{u}^k = \tilde{\mathcal{D}}_i \tilde{u}^{ki}$ (ther is no summation over i) where $\tilde{u}^{ki} = \int \tilde{u}^k d\tilde{x}^i$. We have:

$$u^{ki} + \epsilon \eta^{ki} = \int (u^k + \epsilon \eta^k)(dx^i + \epsilon \, \xi^i_j \, dx^j)$$

$$\tilde{\mathcal{D}}_i (u^{ki} + \epsilon \eta^{ki}) = \tilde{\mathcal{D}}_i \int (u^k dx^i + \epsilon \eta^k dx^i + \epsilon u^k \xi^i_j \, dx^j)$$

or

$$\eta^{ki} = \int (\eta^k + u^k \xi^i_i) \, dx^i \quad (10)$$

Formula (10) is the analogue of formula (5). In order to derive the formulae of the prolongations of transformations (7) for the integrals of arbitrary functions, it suffices to substitute the expansion $\breve{\Phi}$ in a series in the right-hand part of the formula

$$\int d\tilde{x}^i \tilde{\Phi} = \int (\tilde{\Phi} \, dx^i + \epsilon \, \tilde{\Phi} \, \xi^i_j \, dx^j) \quad (11)$$

and to take into account the terms of the first order on ϵ. The higher prolongations can be found in a similar manner.

The integrals in formulae (9), (10), (11) should be treat-

* As previously, the brackets indicate the operator character of the expressions.

ed as line integrals. The independence of these integrals on the path of integration, as the standard calculations show, is ensured by the symmetry of highest prolonged variables over the lower indexes. Taking into account this circumstance, it is ready to show that, for example, the operators (3) and (9) are, as it should be, inverse.

Thus, the integro-differential prolongations are constructed with the help of two mutually-inverse operators. Therefore, to the prolonged variables in whose definition a pair of inverse operators is present and to the variables where this pair is absent at all, the same coordinate should be put in correspondence.

Finally, note the following. Doubling the appropriate space, as it was done in [1] , the constructed groups of symmetry may also be treated as groups conserving the tangency of infinite (in the general case) order, namely, to consider them as groups leaving invariant the manifolds given by formulae:

$$d f^{i_1 \dots i_\nu}_{j_1 \dots j_\mu} - f^{i_1 \dots i_\nu}_{j_1 \dots j_\mu j} \, dx^j = 0$$

$(\mu, \nu = 0, 1, 2, \dots)$ where f are the arbitrary functions of z^i.

V. On symmetries and conservation laws

In consideration of the symmetry groups of Lie-Backlund type the Ibragimov theorem [4] seems to be useful which may be formulated as follows:

The differential consequencies of an equation have a symmetry not less than the symmetry of the initial equation.

For proof, let us use formula (3), rewriting it in the form:*

* Note that $\delta\omega \equiv \epsilon \hat{X} \omega$.

$$\delta \mathcal{D}_i \equiv \tilde{\mathcal{D}}_i - \mathcal{D}_i = - \epsilon \xi^j_i \mathcal{D}_j$$

Let $\hat{X}\omega = 0$ for the equation $\omega = 0$. Then taking into account that $\mathcal{D}_i \omega = 0$ and using the relation

$$\delta(\mathcal{D}_i \omega) = (\delta \mathcal{D}_i)\omega + \mathcal{D}_i \delta\omega$$

or:

$$\mathcal{D}_i \hat{X} \omega - \hat{X} \mathcal{D}_i \omega = \xi^j_i \mathcal{D}_j \omega \tag{12}$$

one gets $\hat{X} \mathcal{D}_i \omega = 0$. Thus, the theorem is proved.*

Let us determine the connection between the symmetry of an equation and that of its integral consequence. Following from the relation:

$$\delta \int \omega \, dx^i = \int (\omega \, dx^i + \delta\omega \, dx^i + \omega \, \delta dx^i) - \int \omega \, dx^i$$

one gets

$$\epsilon \hat{X} \int \omega \, dx^i = \int (\omega \, dx^i + \epsilon \hat{X} \omega \, dx^i + \epsilon \omega \xi^i_j \, dx^j) - \int \omega \, dx^i$$

Then, from $\omega = 0$ and $\hat{X}\omega = 0$ one finds $\hat{X} \int \omega \, dx^i = 0$. Inversely, from $\omega = 0$ and $\hat{X} \int \omega \, dx^i = 0$ it follows $\hat{X}\omega = 0$. Hence, we have shown that the symmetry of the integral consequence of an equation coincides with that of the initial equation.

Whence, the differential consequences and the initial equation have the same symmetry (see eq.(12)).

In ref. [4] for the set of differential equations it was introduced the notion of a relatively G-invariant weak Lagrangian $\mathcal{L}$ as a function of dependent and independent variables such that:

$$\hat{X}\mathcal{L} + \mathcal{L} \xi^i_i = 0$$

$$\frac{\delta \mathcal{L}}{\delta u^k} = 0 \quad (k = 1, \dots, M) \tag{13}$$

* From $\hat{X} \mathcal{D}_i \omega = 0$ it follows only that $\hat{X}\omega = const$

Here $\frac{\delta}{\delta u^\kappa}$ is the variational derivatives. Equalities (13) are fulfilled due to arbitrary differential consequences of the initial set of equations.

For the more general groups of transformations considered in the foregoing section, eqs.(13) must be fulfilled in virtue of integro-differential consequences of the initial system.

The theorem 2 of ref. [4] for the <u>local</u> relatively G-invariant weak Lagrangians is applicable to the case of non-local groups of transformations with very small variation, and namely, the following theorem takes place:

Let the arbitrary set of differential equations with smooth coefficients be given. For this system there exists the conservation law $\mathcal{D}_i A^i = 0$ (due to arbitrary integro-differential consequences) iff the initial system admits a group of non-local transformations and also the relatively G-invariant weak Lagrangian $\mathcal{L}$ of the system exists. The vector A^i is calculated by the formula:

$$A^i = \mathcal{L}\xi^i + (\eta^\alpha - \xi^\ell u^\alpha_\ell)\Big(\frac{\partial \mathcal{L}}{\partial u^\alpha_i} + \sum_{\nu=1}^{\infty}(-1)^\nu \mathcal{D}_{j_1}\cdots\mathcal{D}_{j_\nu}\frac{\partial\mathcal{L}}{\partial u^\alpha_{ij_1\cdots j_\nu}}\Big) +$$

$$+\sum_{s=1}^{\infty}\mathcal{D}_{k_1}\cdots\mathcal{D}_{k_s}(\eta^\alpha - \xi^\ell u^\alpha_\ell)\Big(\frac{\partial\mathcal{L}}{\partial u^\alpha_{ik_1\cdots k_s}} + \sum_{\nu=1}^{\infty}(-1)^\nu\mathcal{D}_{j_1}\cdots\mathcal{D}_{j_\nu}\frac{\partial\mathcal{L}}{\partial u^\alpha_{ij_1\cdots j_\nu k_1\cdots k_s}}\Big)$$

It is obvious that in general case the conserved quantities will be non-local since ξ^i and η^κ are the non-local functions of $x^i, u^\kappa, u^\kappa_{i\cdots}$. However, all the local conservation laws are derived by the given formula also. In this connection, the formulated theorem is more general compared to theorem 2 of ref. [4]. The proof is not, in essence, differ from that of theorem 2 of ref. [4] and it is here omitted.

VI. Conclusion

The above-constructed formalism is a natural scheme for studying the groups of symmetry more general than the Lie-Backlund groups. The groups of this type - the groups of non-local transformations, and correspondingly, the non-local conversed quantities (integrals of motion) are characteristic to the differential equations integrable by the method of inverse spectral transform.

The highest integrals of motion of these equations, both the local and non-local ones, admit a natural interpretation within the framework of the formalism constructed.

This question and also the question on a connection of conservation laws with the group of symmetry irrespective to the variational principle will be considered elsewhere.

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